Cover Sheet – Part I

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Name__________________________________________________________
Part 1: Applied Analysis

Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

(1) Given \( w \in C[0, 1], \) with \( w(x) > 0 \) on \( [0, 1], \) let \( L_w^2[0, 1] \) be the weighted Hilbert space with the inner product

\[
\langle f, g \rangle_w = \int_0^1 f(x)g(x)w(x)dx,
\]

where \( f, g \) are in \( L^2[0, 1]. \) In addition, let \( \{\phi_n(x)\}_{n=0}^\infty \) be the set of orthogonal polynomials generated by using the Gram-Schmidt process on \( \{1, x, x^2 \ldots \} \) in the inner product for \( L_w^2. \) Assume that \( \phi_n(x) = x^n + \text{lower powers}. \)

(a) State the Weierstrass Approximation Theorem and briefly sketch its proof. (Use no more than a page or so.)

(b) You are given that \( C[0, 1] \) is dense in \( L_w^2[0, 1]. \) Show that the orthogonal polynomials \( \{\phi_n(x)\}_{n=0}^\infty \) form a complete, orthogonal set in \( L_w^2. \)

(2) Consider the differential operator \( Lu(x) = -((x + 1)u')', \) with \( x \in [0, 1]. \)

(a) Show that if \( D(L) := \{u \in L^2 \mid Lu \in L^2 \text{ and } u(0) = 0 = u'(1)\}, \) then \( L \) is self adjoint and positive definite.

(b) Find the Green’s function for \( L \) having the domain \( D(L) \) above.

(c) Briefly explain why the eigenfunctions this operator are complete in \( L^2[0, 1]. \)

(3) In the problem below, use the Fourier transform conventions

\[
\mathcal{F}[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx
\]

\[
\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega.
\]

As usual, \( \hat{f} = \mathcal{F}[f]. \)

(a) Show \( \mathcal{F}^4 = I. \) (Hint: \( \mathcal{F}[f(x)] = \mathcal{F}^{-1}[f(-x)].\))

(b) You are given that the equation \( -u''_n + x^2 u_n = (2n + 1)u_n \) has, up to a constant multiple, a unique solution \( u_n \in L^2(\mathbb{R}), \) for \( n = 0, 1, \ldots. \) (You may assume that the solution is smooth enough and decays fast enough to be in Schwartz space.) Show that \( u_n \) is an eigenfunction of the Fourier transform; that is, \( \hat{u}_n(\omega) = \lambda_n u_n(\omega). \) Also, show that \( \lambda_n^4 = 1. \)

(4) Let \( k(x, y) = x^4 y^{12} \) and consider the operator \( Ku(x) = \int_0^1 k(x, y)u(y)dy. \)

(a) Show that \( K \) is a Hilbert-Schmidt operator and that \( \|K\|_{op} \leq \frac{1}{10} \).

(b) State the Fredholm Alternative for the operator \( L = I - \lambda K. \) Explain why it applies in this case. Find all values of \( \lambda \) such that \( Lu = f \) has a unique solution for all \( f \in L^2[0, 1]. \)

(c) Use a Neumann series to find the resolvent \( (I - \lambda K)^{-1} \) for \( \lambda \) small. Sum the series to find the resolvent.
Cover Sheet – Part II

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Name

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Part 2: Numerical Analysis

**Instructions:** Do all problems in this part of the exam. Show all of your work clearly.

**Problem 1:** Consider the following two-points boundary value second order problem in 1-D: Find a function $u$ defined a.e. in $]0, 1[$ such that

$$
-(xK(x)u'(x))' + xq(x)u(x) = xf(x) \text{ a.e. in } ]0, 1[,
$$

where $K \in C^1([0, 1]), q \in C^0([0, 1])$ and $f \in L^2(0, 1)$ are given functions. Assume that there exists a constant $\kappa_0 > 0$ such that $K(x) \geq \kappa_0$ and $q(x) \geq 0$ for all $x \in [0, 1]$. Let $V = \{v \in L^2_{loc}(0, 1) : \sqrt{x}v \in L^2(0, 1), \sqrt{x}v' \in L^2(0, 1)\}$.

Accept as a fact that $V$ is a Hilbert space for the norm

$$
\|v\|_V = \left(\|\sqrt{x}v\|_{L^2(0, 1)}^2 + \|\sqrt{x}v'\|_{L^2(0, 1)}^2\right)^{1/2},
$$

and $C^1([0, 1])$ is dense in $V$ for this norm.

1. Derive the variational formulation (also called weak formulation) of problem (1) in the space $V$.
2. Prove that the corresponding bilinear form of this variational formulation is elliptic (or coercive) in $V$.
   **Hint.** First show that all functions $v$ of $C^1([0, 1])$ satisfy
   $$
   \int_0^1 v(x)^2 dx = v^2(1) - 2 \int_0^1 xv(x)v'(x)dx
   $$
   and then establish the following variant of Poincaré’s inequality
   $$
   \forall v \in V, \|\sqrt{x}v\|_{L^2(0, 1)} \leq \alpha \left(v^2(1) + \|\sqrt{x}v'\|_{L^2(0, 1)}^2\right)^{1/2}
   $$
   for some constant $\alpha > 0$. Based on this equality deduct the ellipticity.
3. Choose an integer $N \geq 2$, set $h = 1/N$, let $x_i = ih$, $0 \leq i \leq N$ and define the finite element space
   $$
   V_h = \{v_h \in C^0([0, 1]) : v_h|_{x_i,x_{i+1}} \in P_1, 0 \leq i \leq N - 1\}.
   $$

Show that $V_h$ is a subspace of $V$. Discretize the variational problem in this space. Prove existence and uniqueness of the discrete solution and establish an error estimate without estimating the norms of the interpolation errors.

**Problem 2:** Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial \Omega$. Let $H^1_0(\Omega) = \{v \in H^1(\Omega) : v(x) = 0 \forall x \in \partial \Omega\}$ be the standard Sobolev space of functions defined on $\Omega$ that vanish on the boundary.

In all that follows, $T > 0$ is a given final time, $c > 0$ is a constant, and $u_0 \in C^0(\Omega)$ are given functions. Consider the parabolic equation: Find a function $u$ defined a.e. in $\Omega \times ]0, T[$ solution of

$$
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + cu = 0 \text{ a.e. in } \Omega \times ]0, T[, 
$$

$$
u(x, t) = 0 \text{ a.e. in } \partial \Omega \times ]0, T[, 
$$

$$
u(x, 0) = u_0(x) \text{ a.e. in } \Omega.
$$
Accept as a fact that problem (2) has one and only one solution \( u \) in \( L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \).

Let \( T_h \) be a finite element partition of \( \Omega \) into triangles \( \tau \) of diameter \( h_\tau \leq h \). Further, let
\[
W_h = \{ v_h \in C^0(\overline{\Omega}) ; \forall \tau \in T_h, v_h|_\tau \in P_1, v_h|_{\partial\Omega} = 0 \},
\]
be a finite element space of continuous piece-wise linear functions over \( T_h \).

Consider the fully discrete backward Euler implicit approximation of (2): for a positive integer, set \( k = T/K \), define \( t_n = nk \), \( 0 \leq n \leq K \), and for each \( 0 \leq n \leq K - 1 \), knowing \( u_h^n \in W_h \) find \( u_h^{n+1} \in W_h \) such that
\[
(3) \quad \forall v_h \in W_h, \quad \frac{1}{k} (u_h^{n+1} - u_h^n, v_h) + a(u_h^{n+1}, v_h) = 0, \quad n = 0, 1, \cdots, K, \quad u_h^0 = I_h(u_0).
\]

Here \((\cdot, \cdot)\) is the inner product in \( L^2(\Omega) \), the bilinear form \( a(u_h^{n+1}, v_h) \) comes from the variational formulation of problem (2), and \( I_h \) is the Lagrange interpolation operator in \( W_h \). Write the expression of \( a(u_h^{n+1}, v_h) \).

1. Show that (3) defines a unique function \( u_h^{n+1} \) in \( W_h \).
2. Prove the following stability estimate
\[
(4) \quad \sup_{1 \leq n \leq K} \| u_h^n \|^2_{L^2(\Omega)} + k \sum_{n=1}^{K} |u_h^n|_{H^1(\Omega)}^2 \leq \| u_h^0 \|^2_{L^2(\Omega)}.
\]

3. Also prove the estimate
\[
(5) \quad \sup_{1 \leq n \leq K} |u_h^n|_{H^1(\Omega)} \leq |u_h^0|_{H^1(\Omega)}.
\]

**Problem 3:** Consider the interval \((0,1)\) and the set of continuous functions \( \hat{v} \) defined on \([0,1] \). Let \( \hat{a}_1 = 0, \hat{a}_2 = \frac{1}{3}, \hat{a}_3 = 1 \).

1. Consider the following two sets of degrees of freedom
\[
\Sigma_1 = \{ \hat{v}(\hat{a}_j), j = 1, 2, 3 \} \quad \Sigma_2 = \{ \hat{v}(\hat{a}_1), \hat{v}(\hat{a}_3), \int_0^1 \hat{v}(s)ds \}.
\]

Write down the basis functions of \( P_2 \) (for both sets of degrees of freedom) such that
(a) \( p_i \in P_2, 1 \leq i \leq 3 \), satisfying: \( p_i(\hat{a}_j) = \delta_{i,j}, 1 \leq i, j \leq 3 \) for the set \( \Sigma_1 \);
(b) \( q_i \in P_2, 1 \leq i \leq 3 \), satisfying:
\[
q_i(\hat{a}_j) = \delta_{i,j}, \int_0^1 q_i(s)ds = 0, i = 1, 3, j = 1, 3,
\]
\[
\int_0^1 q_2(s)ds = 1, q_2(\hat{a}_1) = q_2(\hat{a}_3) = 0, \quad \text{for the set } \Sigma_2.
\]

In both cases, write down the FE interpolant \( \hat{\Pi}(\hat{w}) \) of a given function \( \hat{w} \in C^0([0,1]) \).

2. Consider the interval \([a,b]\), let \( F \) map \([0,1]\) onto \([a,b]\), and let \( v \) be given in \( H^3(a,b) \). Define \( \Pi(v) \) by \((\Pi(v)) \circ F = \hat{\Pi}(v \circ F) \). Give the Bramble Hilbert argument to get an estimate in terms of \( h = b - a \) for the error
\[
\| v' - \Pi(v)' \|_{L^2(a,b)}.
\]

Explain how to modify the proof when \( v \) is less regular, e.g. \( v \in H^2(a,b) \).