Determining Projection Constants of Univariate Polynomial Spaces

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Abstract

The long-standing problem of minimal projections is addressed from a computational point of view. Techniques to determine bounds on the projection constants of univariate polynomial spaces are presented. The upper bound, produced by a linear program, and the lower bound, produced by a semidefinite program exploiting the method of moments, are often close enough to deduce the projection constant with reasonable accuracy. The implementation of these programs makes it possible to find the projection constant of several three-dimensional spaces with five digits of accuracy, as well as the projection constants of the spaces of cubic, quartic, and quintic polynomials with four digits of accuracy. Beliefs about uniqueness and shape-preservation of minimal projections are contested along the way.

Key words and phrases: Minimal projection, projection constant, linear programming, semidefinite programming, method of moments.

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1 Introduction

The problem of minimal projections has attracted the attention of approximation theorists for about half a century. The survey of Cheney and Price [3] still provides a fine account on the topic. The fact that the Fourier projection is uniquely minimal from $\mathcal{C}(\mathbb{T})$ onto the space of trigonometric polynomials of degree at most d, derived from Berman–Marcinkiewicz formula, undoubtedly stands as a highlight of the subject, see [12, 4]. But when the focus is put on algebraic rather than trigonometric polynomials, the situation becomes dramatically more complicated. Besides the trivial cases of degree $d = 0$ and $d = 1$, only the case $d = 2$ has been resolved, albeit at the cost of considerable efforts deployed by Chalmers and Metcalf [2]. In fact, traditional analyses may have

reached their limitation for the problem of minimal projections. The present article is our way of advocating a change of philosophy to promote the integration in classical Approximation Theory of modern optimization methods, in particular methods based on moments and positive polynomials, see [11]. Such techniques have already been used for minimal projections [8] and also in the context of constrained approximation [9].

Before presenting the results and insights generated by our approach, let us formalize the problem of interest precisely. Throughout the article, we consider an M-dimensional subspace $\mathcal{U}$ of the space $\mathcal{C}[-1,1]$ of continuous functions defined on the interval $[-1,1]$. Typically, the space $\mathcal{U}$ consists of algebraic polynomials. A projection P from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ is just a linear map from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ that reproduces $\mathcal{U}$, i.e., that satisfies $P(u) = u$ for all $u \in \mathcal{U}$. A minimal projection from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ is a projection P from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ with minimal norm

$$(1) \qquad \|P\|_{\infty\to\infty} = \max_{\|f\|_\infty \le 1} \|P(f)\|_\infty = \max_{x\in[-1,1]} \max_{\|f\|_\infty \le 1} |P(f)(x)|.$$

The projection constant $\lambda(\mathcal{U})$ of $\mathcal{U}$ (relative to $\mathcal{C}[-1,1]$) is the value of the minimum, i.e.,

$$(2) \qquad \lambda(\mathcal{U}) := \min_P \|P\|_{\infty\to\infty} \quad \text{s.to } P \text{ being a projection from } \mathcal{C}[-1,1] \text{ onto } \mathcal{U}.$$

Fixing a basis $(u_1,\ldots,u_M)$ for $\mathcal{U}$, any projection P from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ can be represented by some linear functionals $\eta_1,\ldots,\eta_M$ on $\mathcal{C}[-1,1]$ as

$$(3) \qquad P(f) = \sum_{m=1}^M \eta_m(f)u_m, \qquad \text{with} \quad \eta_m(u_{m'}) = \delta_{m,m'}, \quad m, m' \in [\![1{:}M]\!].$$

Moreover, any bounded linear functional η on $\mathcal{C}[-1,1]$ can be represented by some signed Borel measure μ on $[-1,1]$ as

$$(4) \qquad \eta(f) = \int_{-1}^1 f\,d\mu.$$

It follows easily that the projection constant of $\mathcal{U}$ can also be written as

$$(5) \qquad \lambda(\mathcal{U}) = \inf_{\mu_1,\ldots,\mu_M} \max_{x\in[-1,1]} \int_{-1}^1 d\left|\sum_{m=1}^M u_m(x)\mu_m\right| \quad \text{s.to} \int_{-1}^1 u_{m'}d\mu_m = \delta_{m,m'}, \; m, m' \in [\![1{:}M]\!],$$

where the infimum (in fact, minimum) is taken over all signed Borel measures $\mu_1,\ldots,\mu_M$ on $[-1,1]$. There is of course a difficulty originating from the infinite-dimensionality of the optimization variables $\mu_1,\ldots,\mu_M$, so the optimization program (5) cannot (a priori) be performed exactly.

Our strategy consists in producing computable upper bounds (see Section 2) and lower bounds (see Section 3) that are sufficiently close to determine the value of the projection constant with, say, four or five digits of accuracy (corresponding here to three or four digits after the decimal point). The computational burden can be lighten by exploiting some symmetry properties of minimal projections (see Section 4). Implementing our strategy enables us, for instance, to:

- retrieve numerically the result of [2] for quadratic polynomials, namely

$$(6) \qquad \lambda(\mathcal{P}_2) \approx 1.2201,$$

and allude[1] to the nonuniqueness of minimal projections onto the quadratics;

- determine with five digits of accuracy the projection constants of several other three-dimensional polynomial spaces, and note in passing that

$$(7) \qquad \lambda(\mathrm{span}\{1, x^2, x^3\}) = 1;$$

- determine with four digits of accuracy the projection constants of the spaces of cubic, quatric, and quintic polynomials, namely

$$(8) \qquad \lambda(\mathcal{P}_3) \approx 1.365,$$
$$(9) \qquad \lambda(\mathcal{P}_4) \approx 1.459,$$
$$(10) \qquad \lambda(\mathcal{P}_5) \approx 1.538,$$

and hint[2] that minimal projections onto $\mathcal{P}_d$ do not in general preserve d-convexity, thus disproving a conjecture from [13];

- provide state-of-the-art upper and lower bounds for the projection constants of the spaces of polynomials of degree at most d until $d = 12$.

All of our results can be reproduced by downloading the MATLAB code available on the authors' webpages. The packages CVX and Chebfun are required to execute the code.

2   Computable upper bound

We present in this section a discretization of the problem (5) that leads to a computable upper bound for the projection constant $\lambda(\mathcal{U})$ of a polynomial subspace $\mathcal{U}$ of $\mathcal{C}[-1, 1]$. This involves in fact two discretizations: one for the signed Borel measures $\mu_1, \ldots, \mu_M$ and one for the interval $[-1, 1]$. We start by discretizing the measures.

Proposition 1. Given a basis $(u_1, \ldots, u_M)$ for a subspace $\mathcal{U}$ of $\mathcal{C}[-1, 1]$, if $v_1, \ldots, v_K$ are distinct points in $[-1, 1]$, then

$$(11) \quad \lambda(\mathcal{U}) \leq \min_{A \in \mathbb{R}^{M \times K}} \max_{x \in [-1,1]} \sum_{k=1}^{K} \left| \sum_{m=1}^{M} A_{m,k} u_m(x) \right| \text{ s.to } \sum_{k=1}^{K} A_{m,k} u_{m'}(v_k) = \delta_{m,m'}, \ m, m' \in [\![1{:}M]\!].$$

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[1]We are not being more assertive here because, strictly speaking, our computed minimal projection is not an exact minimal projection.

[2]Same reservation as in footnote 1.

Proof. We obtain an upper bound for $\lambda(\mathcal{U})$ whenever the minimization in (5) is carried over a subset of all signed Borel measures $\mu_1, \ldots, \mu_M$. In particular, we choose here the subset of all linear combinations of Dirac measures at $v_1, \ldots, v_K$, i.e., measures of the form

$$(12) \qquad \mu_m = \sum_{k=1}^{K} A_{m,k} \delta_{v_k}, \qquad m \in [\![1{:}M]\!].$$

Under this restriction, the duality constraints in (5) read, for all $m, m' \in [\![1{:}M]\!]$,

$$(13) \qquad \delta_{m,m'} = \int_{-1}^{1} u_{m'} d\left(\sum_{k=1}^{K} A_{m,k} \delta_{v_k}\right) = \sum_{k=1}^{K} A_{m,k} \int_{-1}^{1} u_{m'} d\delta_{v_k} = \sum_{k=1}^{K} A_{m,k} u_{m'}(v_k),$$

while the integral appearing in the objective function becomes, for a fixed $x \in [-1, 1]$,

$$(14) \quad \int_{-1}^{1} \left|\sum_{m=1}^{M} u_m(x) d\left(\sum_{k=1}^{K} A_{m,k}\delta_{v_k}\right)\right| = \int_{-1}^{1} \left|\sum_{k=1}^{K}\left(\sum_{m=1}^{M} A_{m,k} u_m(x)\right) d\delta_{v_k}\right|$$

$$= \int_{-1}^{1} \sum_{k=1}^{K} \left|\sum_{m=1}^{M} A_{m,k} u_m(x)\right| d\delta_{v_k} = \sum_{k=1}^{K}\left|\sum_{m=1}^{M} A_{m,k} u_m(x)\right|.$$

Taking the expressions (13) and (14) into account yields the upper bound (11). □

Proposition 1 is not directly exploitable due to the presence of the maximum over the interval $[-1, 1]$. We can replace it by a maximum over a discretized grid, as long as we are able to bound the maximum over $[-1, 1]$ by the maximum over this grid. For polynomials, the comparison between the discrete and continuous max-norms is a well-studied topic, especially for equispaced points (see e.g. [7, 5, 14]). But equispaced points are not the most suitable, since the two norms are comparable when the number of points scales quadratically with the degree. In contrast, for zeros of Chebyshev polynomials, the number of points only needs to scale linearly with the degree. Here is a quantitative version of this assertion, which can be found in [7].

Lemma 2. Let $w_1 > \cdots > w_L \in [-1, 1]$ be the Chebyshev zeros given by $w_\ell = \cos(\theta_\ell)$, where $\theta_\ell = \pi(\ell - 1/2)/L$. For any algebraic polynomial p of degree at most d, one has

$$(15) \qquad \max_{x \in [-1,1]} |p(x)| \leq \cos\left(\frac{\pi}{2}\frac{d}{L}\right)^{-1} \max_{\ell \in [\![1{:}L]\!]} |p(w_\ell)|.$$

We are now in a position to derive the awaited computable upper bound for the projection constant.

Proposition 3. Given a basis $(u_1, \ldots, u_M)$ for a subspace $\mathcal{U}$ of $\mathcal{C}[-1, 1]$ consisting of polynomials of degree at most d, let $v_1, \ldots, v_K$ be distinct points in $[-1, 1]$. With $w_1 > \cdots > w_L$ denoting the Chebyshev zeros $w_\ell = \cos(\pi(\ell - 1/2)/L)$ and with $\rho = \cos\left((\pi d)/(2L)\right)^{-1} \geq 1$, one has

$$(16) \quad \lambda(\mathcal{U}) \leq \rho \times \min_{A \in \mathbb{R}^{M \times K}} \max_{\ell \in [\![1{:}L]\!]} \sum_{k=1}^{K} \left|\sum_{m=1}^{M} A_{m,k} u_m(w_\ell)\right| \text{ s.to } \sum_{k=1}^{K} A_{m,k} u_{m'}(v_k) = \delta_{m,m'}, \; m, m' \in [\![1{:}M]\!].$$

Proof. According to Proposition 1, it suffices to show that, for any $A \in \mathbb{R}^{M \times K}$,

$$(17) \qquad \max_{x \in [-1,1]} \sum_{k=1}^{K} \left| \sum_{m=1}^{M} A_{m,k} u_m(x) \right| \leq \rho \max_{\ell \in [\![1:L]\!]} \sum_{k=1}^{K} \left| \sum_{m=1}^{M} A_{m,k} u_m(w_\ell) \right|.$$

For a fixed $x \in [-1, 1]$, we can find signs $\varepsilon_1, \dots, \varepsilon_K \in \{\pm 1\}$ such that

$$(18) \quad \sum_{k=1}^{K} \left| \sum_{m=1}^{M} A_{m,k} u_m(x) \right| = \left| \sum_{k=1}^{K} \varepsilon_k \sum_{m=1}^{M} A_{m,k} u_m(x) \right| \leq \rho \max_{\ell \in [\![1:L]\!]} \left| \sum_{k=1}^{K} \varepsilon_k \sum_{m=1}^{M} A_{m,k} u_m(w_\ell) \right|$$

$$\leq \rho \max_{\ell \in [\![1:L]\!]} \sum_{k=1}^{K} \left| \sum_{m=1}^{M} A_{m,k} u_m(w_\ell) \right|,$$

where Lemma 2 was used for the first inequality in (18). Taking the maximum over $x \in [-1, 1]$ yields the desired inequality (17). $\qquad \square$

We close this section by highlighting how the upper bound from Proposition 3 is effectively computed by solving a linear program. For this purpose, we introduce the collocation matrices $V \in \mathbb{R}^{K \times M}$ and $W \in \mathbb{R}^{L \times M}$ of the basis $(u_1, \dots, u_M)$ at the points $v_1, \dots, v_K$ (usually chosen as equispaced points in $[-1, 1]$) and at the Chebyshev zeros $w_1, \dots, w_L$. These matrices are defined by

$$(19) \qquad V_{k,m} = u_m(v_k) \qquad \text{and} \qquad W_{\ell,m} = u_m(w_\ell), \qquad k \in [\![1:K]\!], \ \ell \in [\![1:L]\!], \ m \in [\![1:M]\!].$$

With this notation, the objective function and the constraints in (16) read, respectively,

$$(20) \qquad \max_{\ell \in [\![1:L]\!]} \sum_{k=1}^{K} |(WA)_{\ell,k}| \qquad \text{and} \qquad (AV)_{m,m'} = \delta_{m,m'}, \quad m, m' \in [\![1:M]\!].$$

Then, introducing slack variables through $B \in \mathbb{R}^{L \times K}$ (such that $|(WA)_{\ell,k}| \leq B_{\ell,k}$ for all ℓ, k) and $c \in \mathbb{R}$ (such that $\sum_{k=1}^{K} B_{\ell,k} \leq c$ for all ℓ), we arrive at the following linear optimization problem, written in an easily implementable form.

Upper bound for the projection constant

Inputs: basis for a space $\mathcal{U}$ of polynomials of degree $\leq d$, parameters K and L.

$$(21) \qquad \lambda(\mathcal{U}) \leq \cos\left(\frac{\pi}{2} \frac{d}{L}\right)^{-1} \times \min_{\substack{A \in \mathbb{R}^{M \times K} \\ B \in \mathbb{R}^{L \times K} \\ c \in \mathbb{R}}} c \qquad \text{s.to} \quad \begin{cases} AV = I_M, \\ -B \leq WA \leq B, \\ B\mathbf{1} \leq c\mathbf{1}, \end{cases}$$

where the matrices V and W are defined in (19).

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3 Computable lower bound

We present in this section a computable lower bound for the projection constant of a polynomial subspace $\mathcal{U}$ of $\mathcal{C}[-1, 1]$. It again involves a discretization of the problem (5), followed by an application of the moment method. This method, based on classical moment problems (see e.g. [16]), characterizes a measure by its sequence of moments, typically via a semidefinite condition. The discrete Hamburger moment problem, for instance, states that, for a sequence $(y_k)_{k \geq 0}$ of real numbers, there exists a nonnegative measure μ on $\mathbb{R}$ such that

$$(22) \qquad \int_{-\infty}^{\infty} x^k d\mu(x) = y_k, \qquad k \geq 0,$$

if and only if an infinite Hankel matrix is positive semidefinite, precisely

$$(23) \qquad \mathrm{Hank}_{\infty}(y) := \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & \cdots \\ y_1 & y_2 & y_3 & \cdot^{\cdot^{\cdot}} & \\ y_2 & y_3 & \cdot^{\cdot^{\cdot}} & & \\ y_3 & \cdot^{\cdot^{\cdot}} & & & \\ \vdots & & & & \end{bmatrix} \succeq 0.$$

We could directly use this characterization to substitute the measures $\mu_1, \ldots, \mu_M$ by their sequences $y_1, \ldots, y_M$ of moments as optimization variables, but we would need to add extra semidefinite conditions ensuring that the measures $\mu_1, \ldots, \mu_M$ are localized on $[-1, 1]$. Instead, we prefer to rely on the discrete trigonometric moment problem, which states[3] that, for a sequence $(y_k)_{k \geq 0}$ of real numbers, there exists a nonnegative measure μ on $[0, \pi]$ such that

$$(24) \qquad \int_0^{\pi} \cos(k\theta) d\mu(\theta) = y_k, \qquad k \geq 0,$$

if and only if an infinite Toeplitz matix is positive semidefinite, precisely

$$(25) \qquad \mathrm{Toep}_{\infty}(y) := \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & \cdots \\ y_1 & y_0 & y_1 & y_2 & \\ y_1 & y_1 & y_0 & y_1 & \cdot^{\cdot^{\cdot}} \\ y_3 & y_2 & y_1 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \\ \vdots & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \end{bmatrix} \succeq 0.$$

There are two reasons explaining our preference: firstly, localization conditions are not necessary; secondly, the Toeplitz structure is numerically more favorable than the Hankel structure.

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[3]The classical statement, found e.g. in [16, Theorem 11.3], concerns sequences of complex numbers indexed by $n \in \mathbb{Z}$ and obtained as $\int_{|z|=1} z^{-n} d\mu(z)$ for some Radon measure on the unit circle. We omit the verification that it implies the statement being made here.

Now, in order to invoke the discrete trigonometric moment problem, we first have to transform the expression of the projection constant given in (5). This is done below.

Lemma 4. Given a basis $(u_1, \ldots, u_M)$ for a subspace $\mathcal{U}$ of $\mathcal{C}[-1, 1]$, if the functions $\widehat{u}_1, \ldots, \widehat{u}_M$ are defined on $[0, \pi]$ by $\widehat{u}_m(\theta) = u_m(\cos(\theta))$, then

$$(26) \qquad \lambda(\mathcal{U}) = \inf_{\mu_1, \ldots, \mu_M} \max_{\theta \in [0, \pi]} \int_0^\pi d \left| \sum_{m=1}^M \widehat{u}_m(\theta) \mu_m \right| \quad \text{s.to} \int_0^\pi \widehat{u}_{m'} d\mu_m = \delta_{m, m'}, \ m, m' \in [\![1{:}M]\!],$$

where the infimum is taken over all signed Borel measures $\mu_1, \ldots, \mu_M$ on $[0, \pi]$.

Proof. Let us consider the subspace $\widehat{\mathcal{U}}$ of $\mathcal{C}[0, \pi]$ defined by $\widehat{\mathcal{U}} = \{u \circ \cos, u \in \mathcal{U}\}$, for which $(\widehat{u}_1, \ldots, \widehat{u}_M)$ is a basis. We readily verify that, if P is a projection from $\mathcal{C}[-1, 1]$ onto $\mathcal{U}$, then

$$(27) \qquad\qquad Q : g \in \mathcal{C}[0, \pi] \mapsto P(g \circ \arccos) \circ \cos \in \widehat{\mathcal{U}}$$

defines a projection from $\mathcal{C}[0, \pi]$ onto $\widehat{\mathcal{U}}$ satisfying $\|Q\|_{\infty \to \infty} = \|P\|_{\infty \to \infty}$. Conversely, we also see that if Q is a projection from $\mathcal{C}[0, \pi]$ onto $\widehat{\mathcal{U}}$, then

$$(28) \qquad\qquad P : f \in \mathcal{C}[-1, 1] \mapsto Q(f \circ \cos) \circ \arccos \in \mathcal{U}$$

defines a projection from $\mathcal{C}[-1, 1]$ onto $\mathcal{U}$ satisfying $\|P\|_{\infty \to \infty} = \|Q\|_{\infty \to \infty}$. This implies that the projection constant of $\mathcal{U}$ (relative to $\mathcal{C}[-1, 1]$) equals the projection constant of $\widehat{\mathcal{U}}$ (relative to $\mathcal{C}[0, \pi]$), i.e.,

$$(29) \qquad \lambda(\mathcal{U}) = \lambda(\widehat{\mathcal{U}}) = \min_Q \|Q\|_{\infty \to \infty} \quad \text{s.to } Q \text{ being a projection from } \mathcal{C}[0, \pi] \text{ onto } \widehat{\mathcal{U}}.$$

The derivation of (26) from (29) is similar to the derivation of (5) from (2). □

Lemma 4 clearly yields a lower bound for the projection constant if we replace the maximum over the interval $[0, \pi]$ by the maximum over a discretization grid $\theta_1, \ldots, \theta_L$ of $[0, \pi]$, i.e., we have

$$(30) \qquad \lambda(\mathcal{U}) \geq \inf_{\mu_1, \ldots, \mu_M} \max_{\ell \in [\![1{:}L]\!]} \int_0^\pi d \left| \sum_{m=1}^M \widehat{u}_m(\theta_\ell) \mu_m \right| \quad \text{s.to} \int_0^\pi \widehat{u}_{m'} d\mu_m = \delta_{m, m'}, \ m, m' \in [\![1{:}M]\!],$$

where the infimum is taken over all signed Borel measures $\mu_1, \ldots, \mu_M$ on $[0, \pi]$. Our next step consists in recasting the latter minimization problem so as to involve only (nonnegative) Borel measures instead of signed Borel measures. Although the following observation may seem obvious,

we make an extra effort to verify it fully.

Lemma 5. Given a basis $(u_1, \ldots, u_M)$ for a subspace $\mathcal{U}$ of $\mathcal{C}[-1, 1]$, let $\widehat{u}_1, \ldots, \widehat{u}_M$ still denote the functions $u_1 \circ \cos, \ldots, u_M \circ \cos$. If $\theta_1, \ldots, \theta_L \in [0, \pi]$, then

$$(31) \quad \lambda(\mathcal{U}) \geq \inf_{\substack{\mu_1^{\pm}, \ldots, \mu_M^{\pm} \\ \nu_1^{\pm}, \ldots, \nu_L^{\pm}}} \max_{\ell \in [\![1:L]\!]} \int_0^\pi \left(d\nu_\ell^+ + d\nu_\ell^- \right) \quad \text{s.to} \int_0^\pi \widehat{u}_{m'}(d\mu_m^+ - d\mu_m^-) = \delta_{m,m'}, \ m, m' \in [\![1:M]\!],$$

$$\sum_{m=1}^M \widehat{u}_m(\theta_\ell)(\mu_m^+ - \mu_m^-) = \nu_\ell^+ - \nu_\ell^-, \ \ell \in [\![1:L]\!],$$

where the infimum is taken over all (nonnegative) Borel measures $\mu_1^{\pm}, \ldots, \mu_M^{\pm}, \nu_1^{\pm}, \ldots, \nu_L^{\pm}$ on $[0, \pi]$.

Proof. Let α be the value of the infimum in (30) and let β be the value of the infimum in (31). To prove that $\alpha \leq \beta$, we consider miminizers $\mu_1^{\pm}, \ldots, \mu_M^{\pm}, \nu_1^{\pm}, \ldots, \nu_L^{\pm}$ for the problem (31). By virtue of the first constraint in (31), the measures $\mu_m^+ - \mu_m^-$, $m \in [\![1:M]\!]$, are feasible for the problem (30), so that, using the second constraint in (31),

$$(32) \qquad \alpha \leq \max_{\ell \in [\![1:L]\!]} \int_0^\pi d \left| \sum_{m=1}^M \widehat{u}_m(\theta_\ell)(\mu_m^+ - \mu_m^-) \right| = \max_{\ell \in [\![1:L]\!]} \int_0^\pi d|\nu_\ell^+ - \nu_\ell^-|$$

$$\leq \max_{\ell \in [\![1:L]\!]} \int_0^\pi (d\nu_\ell^+ + d\nu_\ell^-) = \beta.$$

To prove that $\beta \leq \alpha$, we consider minimizers $\mu_1 \ldots, \mu_M$ for the problem (30). Then we write the Jordan decomposition of each μ_m, $m \in [\![1:M]\!]$, as $\mu_m = \mu_m^+ - \mu_m^-$ for some (nonnegative) Borel measures $\mu_m^{\pm}$ satisfying $\mu_m^+ \perp \mu_m^-$. For each $\ell \in [\![1:L]\!]$, we also write the Jordan decomposition of $\nu_\ell := \sum_{m=1}^M \widehat{u}_m(\theta_\ell)\mu_m$ as $\nu_\ell = \nu_\ell^+ - \nu_\ell^-$ for some (nonnegative) Borel measures $\nu_\ell^{\pm}$ satisfying $\nu_\ell^+ \perp \nu_\ell^-$. In particular, we have $\int_0^\pi d|\nu_\ell| = \int_0^\pi (d\nu_\ell^+ + d\nu_\ell^-)$. Then, noticing that the Borel measures $\mu_1^{\pm}, \ldots, \mu_M^{\pm}, \nu_1^{\pm}, \ldots, \nu_L^{\pm}$ are feasible for the problem (31), we obtain

$$(33) \qquad \beta \leq \max_{\ell \in [\![1:L]\!]} \int_0^\pi (d\nu_\ell^+ + d\nu_\ell^-) = \max_{\ell \in [\![1:L]\!]} \int_0^\pi d|\nu_\ell| = \max_{\ell \in [\![1:L]\!]} \int_0^\pi d \left| \sum_{m=1}^M \widehat{u}_m(\theta_\ell)\mu_m \right| = \alpha.$$

The proof is now complete. □

The expression in (31) is still not directly exploitable due to the infinite-dimensionality of the Borel measures. We resolve this issue by substituting these measures by their sequences of moments and then by truncating the moment constraints. With fewer constraints, a smaller value for the minimum is produced. At the same time, since the moments discarded in the constraints do not occur in the objective function either, they can be removed altogether to create a finite-dimensional semidefinite program. We make all of this precise in the proof of the following result, which presents

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the awaited computable lower bound. Below, the notation $\text{Toep}_S(y)$ stands for the $S \times S$ Toeplitz matrix constructed from the first S components of a sequence $(y_k)_{k \geq 0}$, i.e.,

(34)
$$
\text{Toep}_S(y) = \begin{bmatrix} y_0 & y_1 & \cdots & \cdots & y_{S-1} \\ y_1 & y_0 & y_1 & & \vdots \\ \vdots & y_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & y_1 \\ y_{S-1} & \cdots & \cdots & y_1 & y_0 \end{bmatrix}.
$$

Proposition 6. Let $\mathcal{U}$ be a subspace of $\mathcal{C}[-1, 1]$ consisting of polynomials of degree at most d. Let $(u_1, \ldots, u_M)$ be a basis for $\mathcal{U}$, whose elements have the Chebyshev expansions

(35)
$$
u_m = \sum_{k=0}^{d} U_{k,m} T_k, \qquad m \in [\![1{:}M]\!].
$$

For an integer $S > d$ and for points $\theta_1, \ldots, \theta_L \in [0, \pi]$, one has

(36) $\quad \lambda(\mathcal{U}) \geq \inf\limits_{\substack{y_1^\pm, \ldots, y_M^\pm \\ z_1^\pm, \ldots, z_L^\pm}} \max\limits_{\ell \in [\![1{:}L]\!]} (z_{\ell,0}^+ + z_{\ell,0}^-) \quad$ s.to $\sum\limits_{k=0}^{d} U_{k,m'}(y_{m,k}^+ - y_{m,k}^-) = \delta_{m,m'}, \ m, m' \in [\![1{:}M]\!],$

$$
\sum_{m=1}^{M} u_m(\cos(\theta_\ell))(y_m^+ - y_m^-) = z_\ell^+ - z_\ell^-, \ \ell \in [\![1{:}L]\!],
$$
$$
\text{Toep}_S(y_m^\pm) \succeq 0, \ m \in [\![1{:}M]\!],
$$
$$
\text{Toep}_S(z_\ell^\pm) \succeq 0, \ \ell \in [\![1{:}L]\!],
$$

where the infimum is taken over all vectors $y_1^\pm, \ldots, y_M^\pm, z_1^\pm, \ldots, z_L^\pm \in \mathbb{R}^S$ indexed from 0 to $S - 1$.

Proof. In the optimization program (31), we substitute the Borel measures $\mu_1^\pm, \ldots, \mu_M^\pm, \nu_1^\pm, \ldots, \nu_L^\pm$ by their infinite sequences $y_1^\pm, \ldots, y_M^\pm, z_1^\pm, \ldots, z_L^\pm$ of moments, identified as

(37)
$$
y_{m,k}^\pm = \int_0^\pi \cos(k\theta) d\mu_m^\pm(\theta), \qquad z_{\ell,k}^\pm = \int_0^\pi \cos(k\theta) d\nu_\ell^\pm(\theta), \qquad k \geq 0,
$$

to reach an equivalent optimization program featuring the objective function $\max_{\ell \in [\![1{:}L]\!]}(z_{\ell,0}^+ + z_{\ell,0}^-)$. As for the constraints (duality, consistency, moments), the first constraint in (31) becomes the first constraint in (36) by virtue of

(38)
$$
\int_0^\pi \widehat{u}_{m'}(\theta)(d\mu_m^+(\theta) - d\mu_m^-(\theta)) = \int_0^\pi \sum_{k=0}^{d} U_{k,m'} T_k(\cos(\theta))(d\mu_m^+(\theta) - d\mu_m^-(\theta))
$$
$$
= \sum_{k=0}^{d} U_{k,m'} \int_0^\pi \cos(k\theta)(d\mu_m^+(\theta) - d\mu_m^-(\theta))
$$
$$
= \sum_{k=0}^{d} U_{k,m'}(y_{m,k}^+ - y_{m,k}^-);
$$

9

the second constraint in (31) is clearly equivalent to the second constraint in (36); while the fact that we are dealing with sequences of moments is reflected by the semidefinite conditions $\mathrm{Toep}_\infty(y_m^\pm) \succeq 0$, $m \in [\![1\!:\!M]\!]$, and $\mathrm{Toep}_\infty(z_\ell^\pm) \succeq 0$, $\ell \in [\![1\!:\!L]\!]$. We now relax the last two sets of constraints by simply imposing, for $\ell \in [\![1\!:\!L]\!]$,

$$\text{(39)} \qquad \sum_{m=1}^{M} u_m(\cos(\theta_\ell))(y_{m,k}^+ - y_{m,k}^-) = z_{\ell,k}^+ - z_{\ell,k}^-, \qquad k \in [\![0\!:\!S\!-\!1]\!],$$

as well as, for $m \in [\![1\!:\!M]\!]$ and $\ell \in [\![1\!:\!L]\!]$,

$$\text{(40)} \qquad \mathrm{Toep}_S(y_m^\pm) \succeq 0 \qquad \text{and} \qquad \mathrm{Toep}_S(z_\ell^\pm) \succeq 0.$$

This leads to a smaller minimum value for the optimization program. And since the relaxed program only involves moments up to order $S - 1$, we can restrict the minimization to the finite sequences $(y_{m,k}^\pm)_{k=0}^{S-1}$, $m \in [\![1\!:\!M]\!]$, and $(z_{\ell,k}^\pm)_{k=0}^{S-1}$, $\ell \in [\![1\!:\!L]\!]$, hence yielding the computable lower bound stated in (36). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We close this section by highlighting how the lower bound from Proposition 6 is expressed as a semidefinite program. For this purpose, besides the matrix $U \in \mathbb{R}^{(d+1)\times M}$ containing the coefficients of $u_1, \ldots, u_M$ in the Chebyshev system $(T_0, \ldots, T_d)$, as indicated in (35), we also consider the collocation matrix $W \in \mathbb{R}^{L\times M}$ with entries

$$\text{(41)} \qquad W_{\ell,m} = u_m(\cos(\theta_\ell)), \qquad \ell \in [\![1\!:\!L]\!],\ m \in [\![1\!:\!M]\!].$$

By further introducing matrices $Y^\pm \in \mathbb{R}^{M\times S}$ with rows $y_1^\pm, \ldots, y_M^\pm \in \mathbb{R}^S$ and $Z^\pm \in \mathbb{R}^{L\times S}$ with rows $z_1^\pm, \ldots, z_L^\pm \in \mathbb{R}^S$, as well as a slack variable $c \in \mathbb{R}$ (such that $z_{\ell,0}^+ + z_{\ell,0}^- \leq c$ for all ℓ), the optimization program in (36) takes the easily implementable form below (where some convenient MATLAB notation is used).[4]

Lower bound for the projection constant

Inputs: basis for a space $\mathcal{U}$ of polynomials of degree $\leq d$, parameters $S > d$ and L.

$$\text{(42)} \qquad \lambda(\mathcal{U}) \geq \min_{\substack{Y^\pm \in \mathbb{R}^{M\times S} \\ Z^\pm \in \mathbb{R}^{L\times S} \\ c \in \mathbb{R}}} c \quad \text{s.to} \quad \begin{cases} (Y^+(:,1\!:\!d+1) - Y^-(:,1\!:\!d+1))\,U = I_M, \\ W\,(Y^+ - Y^-) = Z^+ - Z^-, \\ \mathrm{Toep}_S(Y^\pm(m,:)) \succeq 0, \quad m \in [\![1\!:\!M]\!], \\ \mathrm{Toep}_S(Z^\pm(\ell,:)) \succeq 0, \quad \ell \in [\![1\!:\!L]\!], \\ Z^+(:,1) + Z^-(:,1) \leq c, \end{cases}$$

where the matrices U and V are defined in (35) and (41).

---

[4]The grid points $\cos(\theta_1), \ldots, \cos(\theta_L)$ could be added as inputs — by default, we chose them to be Chebyshev zeros.

4 Exploiting the symmetry of minimal projections

The programs highlighted in (21) and (42) are computationally demanding for large values of the parameters K, L, and S, so any property that can reduce their complexity should be exploited. We shall capitalize on a certain symmetry of minimal projections. To this end, we assume from now on that the subspace $\mathcal{U}$ of $\mathcal{C}[-1,1]$ satisfies

$$(43) \qquad u \in \mathcal{U} \Longrightarrow u(-\cdot) \in \mathcal{U},$$

where $u(-\cdot)$ evidently denotes the function $x \in [-1,1] \mapsto u(-x) \in \mathbb{R}$. Under this assumption, it is known that there exists a minimal projection P from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ which is symmetric, in the sense that

$$(44) \qquad P(f(-\cdot)) = (P(f))(-\cdot) \qquad \text{for all } f \in \mathcal{C}[-1,1].$$

This fact has the following implication whenever assumption (43) holds.[5]

Proposition 7. Let $\mathcal{U}$ be a subspace of $\mathcal{C}[-1,1]$ and let $(u_1^{\mathrm{e}}, \dots, u_{M_{\mathrm{e}}}^{\mathrm{e}}, u_1^{\mathrm{o}}, \dots, u_{M_{\mathrm{o}}}^{\mathrm{o}})$ be a basis for $\mathcal{U}$ arranged in such a way that the u_m^{e} are even functions and the u_m^{o} are odd functions. Considering functions $\widetilde{u}_1^{\mathrm{e}}, \dots, \widetilde{u}_{M_{\mathrm{e}}}^{\mathrm{e}}, \widetilde{u}_1^{\mathrm{o}}, \dots, \widetilde{u}_{M_{\mathrm{o}}}^{\mathrm{o}}$ defined on $[-1,1]$ by

$$(45) \qquad \widetilde{u}_m^{\mathrm{e/o}}(t) := u_m^{\mathrm{e/o}}\left(\frac{t+1}{2}\right), \qquad t \in [-1,1],$$

the projection constant of $\mathcal{U}$ can be expressed as

$$(46) \quad \lambda(\mathcal{U}) = \inf_{\substack{\widetilde{\mu}_1^{\mathrm{e}}, \dots, \widetilde{\mu}_{M_{\mathrm{e}}}^{\mathrm{e}} \\ \widetilde{\mu}_1^{\mathrm{o}}, \dots, \widetilde{\mu}_{M_{\mathrm{o}}}^{\mathrm{o}}}} \max_{x \in [0,1]} \int_{-1}^{1} d\max\left\{\left|\sum_{m=1}^{M_{\mathrm{e}}} u_m^{\mathrm{e}}(x)\widetilde{\mu}_m^{\mathrm{e}}\right|, \left|\sum_{m=1}^{M_{\mathrm{o}}} u_m^{\mathrm{o}}(x)\widetilde{\mu}_m^{\mathrm{o}}\right|\right\}$$

$$\text{s.to} \int_{-1}^{1} \widetilde{u}_{m'}^{\mathrm{e}} d\widetilde{\mu}_m^{\mathrm{e}} = \delta_{m,m'}, \ m,m' \in [\![1{:}M_{\mathrm{e}}]\!], \qquad \int_{-1}^{1} \widetilde{u}_{m'}^{\mathrm{o}} d\widetilde{\mu}_m^{\mathrm{o}} = \delta_{m,m'}, \ m,m' \in [\![1{:}M_{\mathrm{o}}]\!],$$

where the infimum is taken over all signed Borel measures $\widetilde{\mu}_1^{\mathrm{e}}, \dots, \widetilde{\mu}_{M_{\mathrm{e}}}^{\mathrm{e}}, \widetilde{\mu}_1^{\mathrm{o}}, \dots, \widetilde{\mu}_{M_{\mathrm{o}}}^{\mathrm{o}}$ on $[-1,1]$.

Proof. Let us consider a symmetric projection from $\mathcal{C}[-1,1]$ onto $\mathcal{U}$ written as

$$(47) \qquad P(f) = \sum_{m=1}^{M_{\mathrm{e}}} \eta_m^{\mathrm{e}}(f)u_m^{\mathrm{e}} + \sum_{m=1}^{M_{\mathrm{o}}} \eta_m^{\mathrm{o}}(f)u_m^{\mathrm{o}}, \qquad f \in \mathcal{C}[-1,1].$$

The condition (44) is readily seen to be equivalent to the conditions

$$(48) \qquad \eta_m^{\mathrm{e}}(f(-\cdot)) = \eta_m^{\mathrm{e}}(f), \qquad \eta_m^{\mathrm{o}}(f(-\cdot)) = -\eta_m^{\mathrm{o}}(f), \qquad f \in \mathcal{C}[-1,1],$$

---

[5]Under assumption (43), one can verify that there exists a basis $(u_1, \dots, u_M)$ whose elements are either even or odd functions (verify, for instance, that $\mathcal{U} = \mathcal{U}^{\mathrm{e}} \overset{\perp}{\oplus} \mathcal{U}^{\mathrm{o}}$, where $\mathcal{U}^{\mathrm{e/o}} := \{u \in \mathcal{U} : u \text{ is an even/odd function}\}$, and concatenate a basis for $\mathcal{U}^{\mathrm{e}}$ with a basis for $\mathcal{U}^{\mathrm{o}}$).

which in turn are equivalent, in terms of measures $\mu_1^{\mathrm{e}}, \ldots, \mu_{M_{\mathrm{e}}}^{\mathrm{e}}, \mu_1^{\mathrm{o}}, \ldots, \mu_{M_{\mathrm{o}}}^{\mathrm{o}}$ representing the linear functionals $\eta_1^{\mathrm{e}}, \ldots, \eta_{M_{\mathrm{e}}}^{\mathrm{e}}, \eta_1^{\mathrm{o}}, \ldots, \eta_{M_{\mathrm{o}}}^{\mathrm{o}}$, to the conditions

$$(49) \qquad d\mu_m^{\mathrm{e}}(-\cdot) = d\mu_m^{\mathrm{e}}, \qquad d\mu_m^{\mathrm{o}}(-\cdot) = -d\mu_m^{\mathrm{o}}.$$

Then, the norm of the projection P satisfies

$$(50) \qquad \begin{aligned} \|P\|_{\infty\to\infty} &= \max_{x\in[-1,1]} \int_{-1}^{1} d \left| \sum_{m=1}^{M_{\mathrm{e}}} u_m^{\mathrm{e}}(x)\mu_m^{\mathrm{e}} + \sum_{m=1}^{M_{\mathrm{o}}} u_m^{\mathrm{o}}(x)\mu_m^{\mathrm{o}} \right| \\ &= \max_{x\in[-1,1]} \int_{-1}^{1} d \left| \sum_{m=1}^{M_{\mathrm{e}}} u_m^{\mathrm{e}}(x)\mu_m^{\mathrm{e}} - \sum_{m=1}^{M_{\mathrm{o}}} u_m^{\mathrm{o}}(x)\mu_m^{\mathrm{o}} \right| \\ &= \max_{x\in[-1,1]} \int_{-1}^{1} d \max\left\{ \left| \sum_{m=1}^{M_{\mathrm{e}}} u_m^{\mathrm{e}}(x)d\mu_m^{\mathrm{e}} \right|, \left| \sum_{m=1}^{M_{\mathrm{o}}} u_m^{\mathrm{o}}(x)d\mu_m^{\mathrm{o}} \right| \right\}. \end{aligned}$$

The last equality follows from the identity $(|a+b| + |a-b|)/2 = \max\{|a|,|b|\}$, which is valid (see [15, page 52, Corollary 1]) in any vector lattice, and hence in the Banach lattice of signed measures equipped with the total variation norm. Noticing the invariance of the expression (50) under the change $x \leftrightarrow -x$ and taking (49) into account, we can further write

$$(51) \qquad \begin{aligned} \|P\|_{\infty\to\infty} &= \max_{x\in[0,1]} 2 \int_0^1 d \max\left\{ \left| \sum_{m=1}^{M_{\mathrm{e}}} u_m^{\mathrm{e}}(x)\mu_m^{\mathrm{e}} \right|, \left| \sum_{m=1}^{M_{\mathrm{o}}} u_m^{\mathrm{o}}(x)\mu_m^{\mathrm{o}} \right| \right\} \\ &= \max_{x\in[0,1]} \int_{-1}^{1} d \max\left\{ \left| \sum_{m=1}^{M_{\mathrm{e}}} u_m^{\mathrm{e}}(x)\widetilde{\mu}_m^{\mathrm{e}} \right|, \left| \sum_{m=1}^{M_{\mathrm{o}}} u_m^{\mathrm{o}}(x)\widetilde{\mu}_m^{\mathrm{o}} \right| \right\}, \end{aligned}$$

where the measures $\widetilde{\mu}_m^{\mathrm{e/o}}$ simply represent the restrictions to $[0,1]$ of the measures $\mu_m^{\mathrm{e/o}}$ that have been transposed to $[-1,1]$, i.e., they are obtained through the identification

$$(52) \qquad d\widetilde{\mu}_m^{\mathrm{e/o}}(t) = 2d\mu_m^{\mathrm{e/o}}(\tau), \qquad t\in[-1,1], \tau\in[0,1] \text{ being linked via } t = 2\tau - 1, \ \tau = \frac{t+1}{2}.$$

Thanks to (49), we can take the $\widetilde{\mu}_m^{\mathrm{e/o}}$ as new optimization variables in the minimization of the norm of a symmetric projection, whose expression (51) is the objective function in (46). We now just have to impose the appropriate duality constraints making P a projection onto $\mathcal{U}$. Among them, the constraints $\eta_m^{\mathrm{e/o}}(u_{m'}^{\mathrm{o/e}}) = 0$ are automatically fulfilled, while the constraints $\eta_m^{\mathrm{e/o}}(u_{m'}^{\mathrm{e/o}}) = \delta_{m,m'}$ reduce to

$$(53) \qquad \delta_{m,m'} = \int_{-1}^{1} u_{m'}^{\mathrm{e/o}} d\mu_m^{\mathrm{e/o}} = 2\int_0^1 u_{m'}^{\mathrm{e/o}}(\tau) d\mu_m^{\mathrm{e/o}}(\tau) = \int_{-1}^{1} \widetilde{u}_{m'}^{\mathrm{e/o}}(t) d\widetilde{\mu}_m^{\mathrm{e/o}}(t).$$

These are indeed the constraints in (46), so the proposition is proved. $\square$

We proceed by highlighting the computable bounds on the projection constant generated by the reformulation (46). Much of the ingredients for deriving these bounds are similar to the ones presented in Sections 2 and 3, so we do not expand on details at all.

4.1 Implication for the upper bound

As in Section 2, we first derive an upper bound for the projection constant by minimizing only over linear combinations

$$
(54) \qquad \widetilde{\mu}_m^{\mathrm{e/o}} = \sum_{k=1}^{K} A_{m,k}^{\mathrm{e/o}} \delta_{v_k}.
$$

of Dirac measures at $v_1, \ldots, v_K \in [-1, 1]$. Then we again discretize by replacing the maximum over $[0, 1]$ by the maximum over a grid $w_1^+ > \cdots > w_L^+$ consisting of positive Chebyshev zeros $w_\ell^+ = \cos(\pi(\ell - 1/2)/(2L))$. Thus, we arrive at the computable upper bound

$$
(55) \qquad \lambda(\mathcal{U}) \leq \rho \times \min_{A^{\mathrm{e/o}} \in \mathbb{R}^{M_{\mathrm{e/o}} \times K}} \max_{\ell \in [\![1:L]\!]} \sum_{k=1}^{K} \max \left\{ \left| \sum_{m=1}^{M_{\mathrm{e}}} A_{m,k}^{\mathrm{e}} u_m^{\mathrm{e}}(w_\ell^+) \right|, \left| \sum_{m=1}^{M_{\mathrm{o}}} A_{m,k}^{\mathrm{o}} u_m^{\mathrm{o}}(w_\ell^+) \right| \right\}
$$
$$
\text{s.to} \ \sum_{k=1}^{K} A_{m,k}^{\mathrm{e/o}} \widetilde{u}_{m'}^{\mathrm{e/o}}(v_k) = \delta_{m,m'}, \, m, m' \in [\![1:M_{\mathrm{e/o}}]\!],
$$

where $\rho = \cos\left((\pi d)/(4L)\right)^{-1}$. To transform the latter into a linear program, we introduce the collocation matrices $V^{\mathrm{e/o}} \in \mathbb{R}^{K \times M_{\mathrm{e/o}}}$ and $W^{\mathrm{e/o}} \in \mathbb{R}^{L \times M_{\mathrm{e/o}}}$ defined by

$$
(56) \qquad V_{k,m}^{\mathrm{e/o}} = \widetilde{u}_m^{\mathrm{e/o}}(v_k) \qquad \text{and} \qquad W_{\ell,m}^{\mathrm{e/o}} = u_m^{\mathrm{e/o}}(w_\ell^+), \qquad k \in [\![1:K]\!], \, \ell \in [\![1:L]\!], \, m \in [\![1:M_{\mathrm{e/o}}]\!],
$$

as well as slack variables through $B \in \mathbb{R}^{L \times K}$ and $c \in \mathbb{R}$. All in all, we obtain the following implementable form of the upper bound.

Upper bound for the projection constant — symmetry exploited

Inputs: basis for a symmetric space $\mathcal{U}$ of polynomials of degree $\leq d$, parameters $S > d$ and L.

$$
(57) \qquad \lambda(\mathcal{U}) \leq \cos\left(\frac{\pi}{4} \frac{d}{L} \right)^{-1} \times \min_{\substack{A^{\mathrm{e/o}} \in \mathbb{R}^{M_{\mathrm{e/o}} \times K} \\ B \in \mathbb{R}^{L \times K} \\ c \in \mathbb{R}}} c \qquad \text{s.to} \quad \begin{cases} A^{\mathrm{e/o}} V^{\mathrm{e/o}} = I_{M_{\mathrm{e/o}}}, \\ -B \leq W^{\mathrm{e/o}} A^{\mathrm{e/o}} \leq B, \\ B\mathbf{1} \leq c\mathbf{1}, \end{cases}
$$

where the matrices $V^{\mathrm{e/o}}$ and $W^{\mathrm{e/o}}$ are defined in (56).

4.2 Implication for the lower bound

As in Section 3, the minimization program (46) is first transformed to make it amenable to the trigonometric moment problem. The reformulation will involve a maximum over $[0, \pi/2]$, which is

lower bounded by the maximum over a grid $\theta_1^+, \ldots, \theta_L^+$. With $w_\ell^+ := \cos(\theta_\ell^+) \in [0, 1]$, we obtain

$$(58) \qquad \lambda(\mathcal{U}) \geq \inf_{\substack{\widetilde{\mu}_1^e, \ldots, \widetilde{\mu}_{M_e}^e \\ \widetilde{\mu}_1^o, \ldots, \widetilde{\mu}_{M_o}^o}} \max_{\ell \in [\![1:L]\!]} \int_0^\pi d \max \left\{ \left| \sum_{m=1}^{M_e} u_m^e(w_\ell^+) \widetilde{\mu}_m^e \right|, \left| \sum_{m=1}^{M_o} u_m^o(w_\ell^+) \widetilde{\mu}_m^o \right| \right\}$$

$$\text{s.to} \int_0^\pi \widetilde{u}_{m'}^{e/o}(\cos(\theta)) d\widetilde{\mu}_m^{e/o}(\theta) = \delta_{m,m'}, \quad m, m' \in [\![1:M_{e/o}]\!],$$

where the infimum is taken over all signed Borel measures $\widetilde{\mu}_1^e, \ldots, \widetilde{\mu}_{M_e}^e, \widetilde{\mu}_1^o, \ldots, \widetilde{\mu}_{M_o}^o$ on $[0, \pi]$. Deviating slightly from the earlier strategy, we now introduce as slack variables some (nonnegative) Borel measures $\nu_1, \ldots, \nu_L$ satisfying, for each $\ell \in [\![1:L]\!]$, the constraint

$$(59) \qquad \nu_\ell \geq \max \left\{ \left| \sum_{m=1}^{M_e} u_m^e(w_\ell^+) \widetilde{\mu}_m^e \right|, \left| \sum_{m=1}^{M_o} u_m^o(w_\ell^+) \widetilde{\mu}_m^o \right| \right\},$$

which implies four constraints written in condensed form as

$$(60) \qquad \nu_\ell \geq \pm \sum_{m=1}^{M_{e/o}} u_m^{e/o}(w_\ell^+) \widetilde{\mu}_m^{e/o}.$$

Writing the Chebyshev expansions of $\widetilde{u}_1^e, \ldots, \widetilde{u}_{M_e}^e, \widetilde{u}_1^o, \ldots, \widetilde{u}_{M_o}^o$ as

$$(61) \qquad \widetilde{u}_m^{e/o} = \sum_{k=0}^d \widetilde{U}_{k,m}^{e/o} T_k, \qquad m \in [\![1:M_{e/o}]\!],$$

and substituting the measures $\widetilde{\mu}_m^{e/o}$ and ν_ℓ by their sequences $y_m^{e/o}$ and z_ℓ of moments yields to

$$(62) \quad \lambda(\mathcal{U}) \geq \inf_{\substack{y_1^{e/o}, \ldots, y_{M_{e/o}}^{e/o} \\ z_1, \ldots, z_L}} \max_{\ell \in [\![1:L]\!]} z_{\ell,0} \quad \text{s.to} \quad \sum_{k=0}^d \widetilde{U}_{k,m'}^{e/o} y_{m,k}^{e/o} = \delta_{m,m'}, \ m, m' \in [\![1:M_{e/o}]\!],$$

$$\text{Toep}_\infty(z_\ell) \succeq \pm\text{Toep}_\infty \left(\sum_{m=1}^{M_{e/o}} u_m^{e/o}(w_\ell^+) y_m^{e/o} \right), \ \ell \in [\![1:L]\!],$$

where the infimum is taken over all infinite sequences $y_1^{e/o}, \ldots, y_{M_{e/o}}^{e/o}, z_1, \ldots, z_L \in \mathbb{R}^{\mathbb{N}}$. The infinite semidefinite constraints are now truncated to a level $S > d$, producing a lower bound involving only the finite sequences of moments $(y_{m,k}^{e/o})_{k=0}^{S-1}$, $m \in [\![1:M_{e/o}]\!]$, and $(z_{\ell,k})_{k=0}^{S-1}$, $\ell \in [\![1:L]\!]$. Finally, in order to state the corresponding semidefinite program in an easily implementable form, we gather these moments in matrices $Y^{e/o} \in \mathbb{R}^{M_{e/o} \times S}$ and $Z \in \mathbb{R}^{L \times S}$, and we define collocation matrices $W^{e/o} \in \mathbb{R}^{L \times M_{e/o}}$ with entries

$$(63) \qquad W_{\ell,m}^{e/o} = u_m^{e/o}(w_\ell^+), \qquad \ell \in [\![1:L]\!], \ m \in [\![1:M_{e/o}]\!].$$

After introducing one last slack variable $c \in \mathbb{R}$, we arrive at the following form of the computable lower bound (where some convenient MATLAB notation is again used).[6]

---

[6]In a similar spirit to footnote 4, the grid points $w_1^+ = \cos(\theta_1), \ldots, w_L^+ = \cos(\theta_L)$ could be added as inputs — by default, we chose them to be positive Chebyshev zeros.

Lower bound for the projection constant — symmetry exploited

Inputs: basis for a symmetric space $\mathcal{U}$ of polynomials of degree $\leq d$, parameters $S > d$ and L.

$$(64) \quad \lambda(\mathcal{U}) \geq \min_{\substack{Y^{\mathrm{e/o}} \in \mathbb{R}^{M_{\mathrm{e/o}} \times S} \\ Z \in \mathbb{R}^{L \times S} \\ c \in \mathbb{R}}} c \quad \text{s.to} \quad \begin{cases} Y^{\mathrm{e/o}}(:, 1{:}d+1)\, \widetilde{U}^{\mathrm{e/o}} = I_{M_{\mathrm{e/o}}}, \\ \mathrm{Toep}_S(Z(\ell, :)) \succeq \pm \mathrm{Toep}_S(W^{\mathrm{e/o}}(\ell, :)Y^{\mathrm{e/o}}),\ \ell \in [\![1{:}L]\!], \\ Z(:, 1) \leq c, \end{cases}$$

where the matrices $\widetilde{U}^{\mathrm{e/o}}$ and $W^{\mathrm{e/o}}$ are defined in (61) and (63).

5 Computational results

This section gives an account of the experiments carried out using our method for specific spaces of univariate polynomials. The experiments can be reproduced by downloading the MATLAB file tied to this paper, available on the authors' webpages. Note that the code relies on CVX [1], a MATLAB package for specifying and solving convex programs, and on Chebfun [6] for its convenience to deal with Chebyshev expansions.

5.1 Validation of the code

In order to certify the correct implementation of the codes computing the upper and lower bounds (57) and (64), we take as a benchmark the inevitable result [2] of Chalmers and Metcalf, who managed to determine analytically the projection constant of the space of quadratic polynomials. They obtained

$$(65) \qquad\qquad \lambda(\mathcal{P}_2) \approx 1.220173064217988\ldots$$

and exhibited a minimal projection given by $P(f)(x) = \sum_{m=1}^{3} \eta_m(f) x^{m-1}$, where

$$(66) \quad \eta_1(f) = \quad Af(-1) + Bf(0) + Af(1) \quad + \int_{I_1} \frac{a_1|t| + b_1}{(1 + w_1|t|)^3} f(t)dt + \int_{I_2} \frac{a_2|t| + b_2}{(1 + w_2|t|)^3} f(t)dt,$$

$$(67) \quad \eta_2(f) = \quad\quad -Cf(-1) + Cf(1) \quad\quad + \int_{I_1} \frac{c_1 t}{(1 + w_1|t|)^3} f(t)dt + \int_{I_2} \frac{c_2 t}{(1 + w_2|t|)^3} f(t)dt,$$

$$(68) \quad \eta_3(f) = \quad Df(-1) - Bf(0) + Df(1) \quad + \int_{I_1} \frac{d_1|t| - b_1}{(1 + w_1|t|)^3} f(t)dt + \int_{I_2} \frac{d_2|t| - b_2}{(1 + w_2|t|)^3} f(t)dt,$$

with $I_1 = [-s_{1,2}, -s_{1,1}] \cup [s_{1,1}, s_{1,2}]$, $I_2 = [-s_{2,2}, -s_{2,1}] \cup [s_{2,1}, s_{2,2}]$, and with parameters A, B, C, D, $a_1, b_1, c_1, d_1,\ a_2, b_2, c_2, d_2,\ w_1, w_2,\ s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}$ determined in the body of [2]. Our code does

allow us to retrieve the value (65) up to five digits of accuracy.[7] Incidentally, our experiment suggests that minimal projections onto the quadratics are not unique, as illustrated by Figure 1 which superimposes the measures associated to (66)-(67)-(68) and the discretized measures obtained by solving (57) — we have removed the atomic parts at $-1, 0, 1$, which were similar. It is worth noticing, nonetheless, that the supports of all the measures seem to be the same.
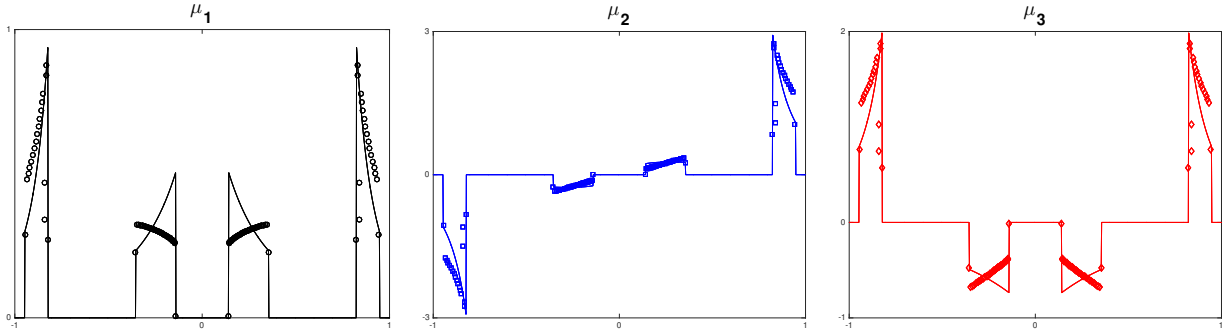

Figure 1: The nonatomic parts of measures μ_1, μ_2, μ_3 associated with the linear functionals η_1, η_2, η_3 found in [2] (continuous lines), together with their discrete approximations obtained as solutions of (57), minus the parts at -1, 0, and 1 (circles, squares, diamonds).

Another way to validate of our code is offered by the three-dimensional space $\mathrm{span}\{1, x^2, x^3\}$, i.e., the subspace $\mathcal{C}[-1, 1]$ spanned by $x \mapsto 1$, $x \mapsto x^2$, and $x \mapsto x^3$. Indeed, its projection constant is exactly equal to one. More generally, for any even integer $a > 0$ and any odd integer $b > 0$, the space $\mathrm{span}\{1, x^a, x^{a+b}\} = \mathrm{span}\{1 - x^a, x^a(1 - x^b), x^a(1 + x^b)\}$ admits a projection of norm one. To see this, we notice that the interpolating projection at the points -1, 0, and 1, which is given by

$$(69) \qquad P(f)(x) = \frac{1}{2}f(-1)x^a(1 - x^b) + f(0)(1 - x^a) + \frac{1}{2}f(1)x^a(1 + x^b),$$

satisfies, for all $f \in \mathcal{C}[-1, 1]$ and $x \in [-1, 1]$,

$$(70) \quad |P(f)(x)| \leq \max\{|f(-1)|, |f(0)|, |f(1)|\} \left(\frac{1}{2}x^a(1 - x^b) + 1 - x^a + \frac{1}{2}x^a(1 + x^b) \right) \leq \|f\|_\infty.$$

Our code confirms the value $\lambda(\mathrm{span}\{1, x^2, x^3\}) = 1$ (not with perfect accuracy, though, which is why we refrain from supplying numerical values with more than five digits in general).

5.2 Other-three dimensional polynomial spaces

The code distributed with the MATLAB reproducible can be effortlessly applied to any univariate polynomial space, so long as it can be executed with parameters large enough for the upper and

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[7]in about five minutes for the upper bound and fifteen minutes for the lower bound, with the capabilities offered by a laptop computer at the time this article was written.

lower bounds to match up to the desired accuracy. Five digits of accuracy can typically be achieved for three-dimensional spaces of moderate degree. The following table summarizes the values of projection constants obtained for several different spaces.[8]

$\mathcal{V}$ spanned by	$\{1, x, x^3\}$	$\{T_0, T_2, T_3\}$	$\{U_0, U_2, U_3\}$	$\{x, x^2, x^3\}$	$\{T_1, T_2, T_3\}$	$\{U_1, U_2, U_3\}$
$\lambda(\mathcal{V})$	≈ 1.4723	≈ 1.4460	≈ 1.1522	≈ 1.3325	≈ 1.4065	≈ 1.2354

Table 1: Projection constants of three-dimensional spaces spanned by monomials and Chebyshev polynomials of the first and second kind.

5.3 Polynomial spaces of higher dimensions

Looking at (57) and (64), we see that the dimension $M = M_{\mathrm{e}} + M_{\mathrm{o}}$ is far from being a factor influencing the computational cost of the optimization programs, so our code can easily be executed for polynomial spaces of dimension higher than three. For instance, we can deal with the spaces of cubic, quartic, and quintic polynomials and compute their projection constants with four digits of accuracy as

$$(71) \qquad \lambda(\mathcal{P}_3) \approx 1.365, \qquad \lambda(\mathcal{P}_4) \approx 1.459, \qquad \lambda(\mathcal{P}_5) \approx 1.538.$$

Obtaining the same accuracy necessitates larger parameters K, L, S when the dimension increases. For degree $d > 5$, with our modest computational investment, we could locate the projection constants of the spaces $\mathcal{P}_d$ of polynomials of degree at most d in the ranges presented in Table 2 below. The improvement with respect to the ranges found in [10] is particularly noticeable for the lower bounds (recall that this is where the method of moments came into the picture).

Let us now come to a close by examining the approximate minimal projection onto the cubics obtained by solving (57). For the measures $\mu_1, \mu_2, \mu_3, \mu_4$ associated with the functionals dual to $1, x, x^2, x^3$, we detected atoms at -1 and 1 (but none at 0) and Figure 2 indicates that their continuous parts seem to possess a common support strictly included in $[-1, 1]$, as was the case for quadratics. But a disparity with the quadratics now occurs in terms of shape preservation. It was conjectured in [13], and proved for $d = 2$, that (one of the) minimal projections onto $\mathcal{P}_d$ preserve d-convexity. This means that if P is a minimal projection from $\mathcal{C}[-1, 1]$ onto $\mathcal{P}_d$, then, for any $f \in \mathcal{C}^d[-1, 1]$,

$$(72) \qquad f^{(d)} \geq 0 \text{ on } [-1, 1] \overset{?}{\Longrightarrow} (P(f))^{(d)} \geq 0 \text{ on } [-1, 1],$$

or equivalently, writing $P(f)(x) = \sum_{m=1}^{d+1} \eta_m(f) x^{m-1}$,

$$(73) \qquad f^{(d)} \geq 0 \text{ on } [-1, 1] \overset{?}{\Longrightarrow} \eta_{d+1}(f) \geq 0.$$

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[8]For the space $\mathrm{span}\{T_1, T_2, T_3\}$, it was more effective to compute the lower bound using a 'symmetrization' of (42) more direct than (64). Its implementation is also included in the reproducible.

$\lambda(\mathcal{P}_d)$	known lower bound	our lower bound	our upper bound	known upper bound
$d=3$	1.3539	1.35667...	1.35696...	1.3577
$d=4$	1.4524	1.45902...	1.45951...	1.4611
$d=5$	1.525	1.53817...	1.53895...	1.543
$d=6$	1.580	1.60271...	1.60383...	1.613
$d=7$	1.624	1.65693...	1.65859...	1.669
$d=8$	1.660	1.70483...	1.70731...	1.721
$d=9$	1.678	1.74774...	1.75107...	1.775
$d=10$	1.696	1.78658...	1.79076...	1.814
$d=11$	NA	1.82169...	1.82701...	NA
$d=12$	NA	1.85380...	1.86216...	NA

Table 2: For the spaces $\mathcal{P}_d$ of polynomials of degree at most d, lower and upper bounds on the projection constants obtained by our method compared to the ones stated in [10].

Our computations for $d=3$ give some insight that this conjecture should be false. Indeed, if a minimal projection is approximated by our solution of (57), which has the form

$$(74) \qquad P(f)(x) = \sum_{m=1}^{4}\left(\sum_{k=-K}^{K} A_{m,k}f_k\right)x^{m-1}, \qquad f_k := f\left(\frac{k}{K}\right),$$

and if the condition $f''' \geq 0$ on $[-1,1]$ is replaced by its discrete version

$$(75) \qquad \Delta_3(f)_k := f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k \geq 0, \qquad k \in [\![-K{:}K-3]\!],$$

then, setting $a = A(4,:) \in \mathbb{R}^{2K-1}$, the question becomes

$$(76) \qquad \min_{f\in\mathbb{R}^{2K-1}} \left\{\langle a,f\rangle \ : \ \Delta_3(f) \geq 0\right\} \overset{?}{\geq} 0.$$

This question is answered negatively by solving a linear program.

6 Outlook

As a concluding message, we reiterate our belief in the usefulness of modern optimization techniques for solving problems in Approximation Theory. Purists will argue that 'solving computationally' is not really solving, but benefits are undeniable for building intuition about the problems at hand. This article demonstrated, for instance, how the method of moments elucidates the problem of minimal projections onto polynomial spaces and it strongly hinted that minimal projections are not unique and do not preserve shape. Our technique can be extended in several directions, as long
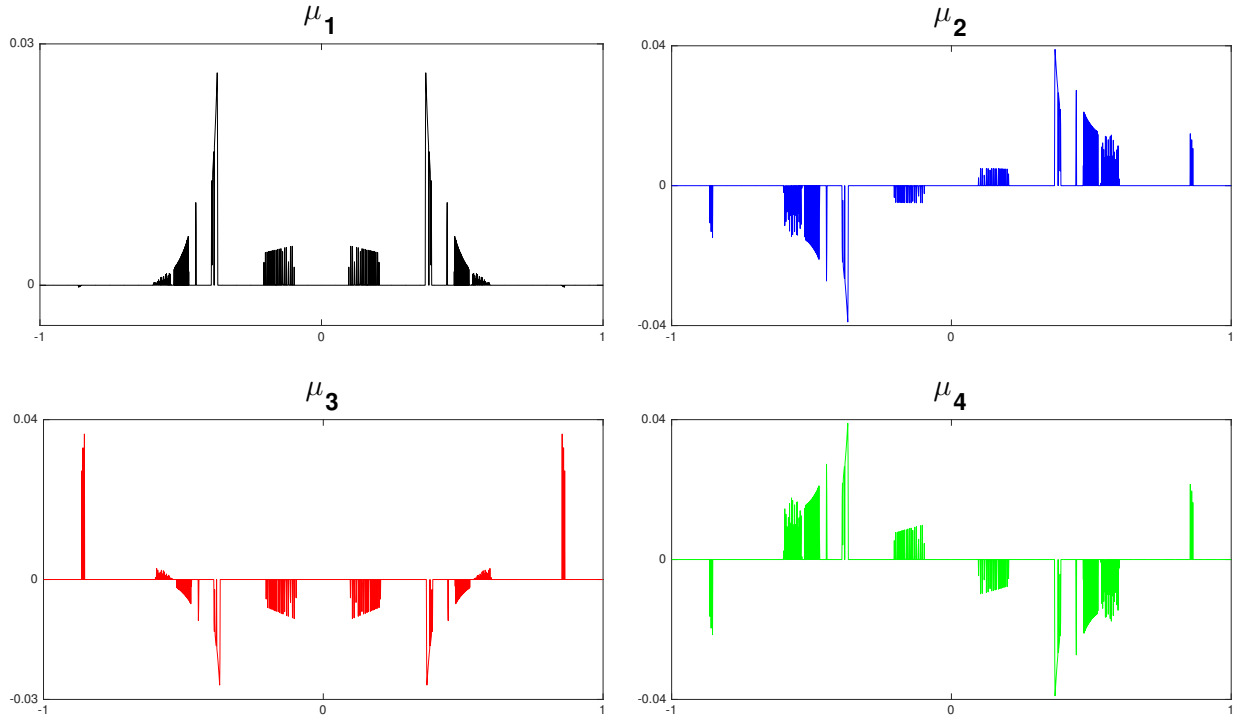
Figure 2: The measures $\mu_1, \mu_2, \mu_3, \mu_4$ (minus atomic parts at -1 and 1) associated with the linear functionals dual to $1, x, x^2, x^3$ in the approximate minimal projection obtained by solving (57).

as linear-programming upper bounds and semidefinite-programming lower bounds match up to a desired accuracy. For example, given subspaces $\mathcal{U} \subseteq \mathcal{V}$ of $\mathcal{C}[-1,1]$, one could find a linear map P from $\mathcal{C}[-1,1]$ into $\mathcal{V}$, not $\mathcal{U}$, with minimal norm among those satisfying $P(u) = u$ for all $u \in \mathcal{U}$, or even satisfying $P(u) = F(u)$ for all $u \in \mathcal{U}$ with some $F \neq \mathrm{Id}_{\mathcal{U}}$, and shape-preservation properties may be added, etc... A particularly interesting situation concerns multivariate polynomial spaces, for which nothing is known except the results of [17]. Conceptually, the same technique applies, but one quickly runs into numerical limitations. One could throw in more computational power, of course, but it seems wiser to refine the method first, possibly with a back-and-forth process between upper bound and lower bound.

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