Math 414
Feb. 23, 2024 (Power ~~outage~~ outage in BLUC.) (1

Last time: Finished uniform convergence
Today: Convergence in the mean + Fourier Transform.

1. ~~Convergence~~ Convergence in the mean — also Thms 1.38 & 1.40 in text

Thm. (HW 3, prob. 2) The partial sum $S_N(x)$ converges in the mean to $f \in L^2[-\pi,\pi)$ (or $L^2(0,2\pi)$, etc.) iff

$$\|f\|^2_{L^2[-\pi,\pi]} = \pi\left(2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right).$$

In complex form, this is

$$\|f\|^2_{L^2(-\pi,\pi)} = 2\pi\left(\sum_{n=-\infty}^{\infty} |\alpha_n|^2\right).$$

Example. $$|x| = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4\cos((2k-1)x)}{\pi(2k-1)^2}.$$

$$\Rightarrow \int_{-\pi}^{\pi} |x|^2 dx = \int_{-\pi}^{\pi} x^2 dx = 2\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^3.$$

$$\Rightarrow \frac{2}{3}\pi^3 = \pi\left(2\cdot\left(\frac{\pi}{2}\right)^2 + \sum_{k=1}^{\infty}\left(\frac{4}{\pi(2k-1)^2}\right)^2\right)$$

$$= \pi\left(\frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4^2}{\pi^2(2k-1)^4}\right)$$

$$\cancel{\Rightarrow \frac{2}{3}\pi^2 = \pi^2}$$

$$\Rightarrow \frac{2}{3}\pi^2 = \frac{\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4^2}{\pi^2(2k-1)^4} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{8\pi^4}{3}$$

$$\cancel{\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{4^2}{\pi^2(2k-1)^4} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{16\pi^4}{6} = \frac{8\pi^4}{3}}$$

Example. $f(x) = e^{2x}, \quad x \in [0, 2\pi]$.

$$d_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2x} \cdot e^{-inx} = \frac{1}{2\pi}\int_0^{2\pi} e^{(2-in)x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(2-in)x}}{2-in}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{(2-in)\cdot 2\pi} - 1}{2-in}\right] = \frac{1}{2\pi}\left(\frac{e^{4\pi}-1}{2-in}\right).$$

$$e^{-2\pi i n} = 1$$

The Complex form of Parseval's Thm

$$\int_0^{2\pi} (e^{2x})^2\,dx = 2\pi\left(\sum_{n=-\infty}^{\infty} \frac{1}{(2\pi)^2}\left|\frac{e^{4\pi}-1}{2-in}\right|^2\right)$$

Abs. val

Note:

(i) $$\left|\frac{e^{2\pi}-1}{2-in}\right|^2 = \frac{(e^{4\pi}-1)^2}{|2-in|^2} = \frac{(e^{4\pi}-1)^2}{4+n^2}.$$

$$|2-in|^2 = 2^2 + n^2$$

(ii) $$\int_0^{2\pi} (e^{2x})^2\,dx = \int_0^{2\pi} e^{4x}\,dx = \frac{e^{4x}}{4}\Big|_0^{2\pi} = \frac{e^{8\pi}-1}{4}$$

$$\therefore \quad \frac{e^{8\pi}-1}{4} = (2\pi)\sum_{n=-\infty}^{\infty} \frac{1}{(2\pi)^2}\,\frac{(e^{4\pi}-1)^2}{4+n^2}$$

$$\frac{e^{8\pi}-1}{4} = \frac{1}{2\pi}(e^{4\pi}-1)^2\left(\sum_{n=-\infty}^{\infty}\frac{1}{4+n^2}\right)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}\frac{1}{4+n^2} = \frac{\pi}{2}\,\frac{(e^{8\pi}-1)}{(e^{4\pi}-1)^2} = \cancel{\frac{\pi}{2}\left(\frac{e^{8\pi}-1}{e^{4}}\right)}$$

$$\sum_{n=-\infty}^{\infty}\frac{1}{4+n^2} = \frac{\pi}{2}\,\frac{(e^{4\pi}+1)(e^{4\pi}-1)}{(e^{4\pi}-1)^2} = \frac{\pi}{2}\,\frac{(e^{4\pi}+1)}{(e^{4\pi}-1)}$$

Parseval's Thm

$$\int_{-\pi}^{\pi}|x|^2\,dx = 2\int_0^{\pi} x^2\,dx = \frac{2\pi^3}{3}.$$

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} 2\pi\left|\frac{i(-1)^n}{n}\right|^2 = \sum_{\substack{n\neq 0\\ n=-\infty}}^{\infty}\frac{2\pi}{n^2},$$

$$\Rightarrow \frac{\pi^2}{3} = \sum_{\substack{n\neq 0\\ n=-\infty}}^{\infty}\frac{1}{n^2}.$$

Note $\displaystyle\sum_{n=-\infty}^{-1}\frac{1}{n^2} \underset{n\to -n}{=} \sum_{m=1}^{\infty}\frac{1}{n^2}.$

$$\Rightarrow \frac{\pi^2}{3} = \sum_{n=-\infty}^{-1}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Pt-wise conv. at $x = \frac{\pi}{2}$.

$$\underbrace{\frac{\pi}{2}}_{x} = \sum_{\substack{n\neq 0\\ n=-\infty}}^{\infty}\frac{i(-1)^n e^{\frac{in\pi}{2}}}{n}$$

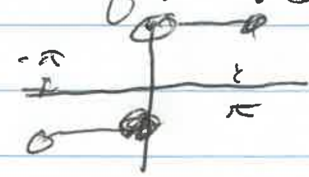
Now $e^{\frac{in\pi}{2}} = \left(e^{i\pi/2}\right)^n = i^n$

$n = 2k$ (even) $i^{2k} = (i^2)^k = (-1)^k$.

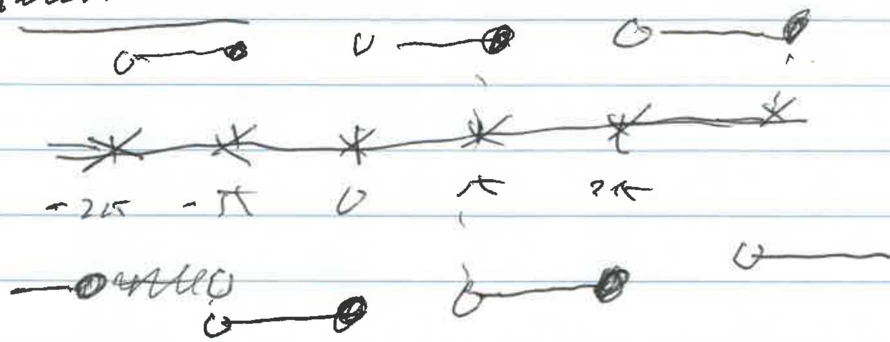
$\therefore\ a_{2k} = \dfrac{i(-1)^{2k}(-1)^{2k}}{2k} = \dfrac{i}{2k}.$

Example Find the complex form of the FS for

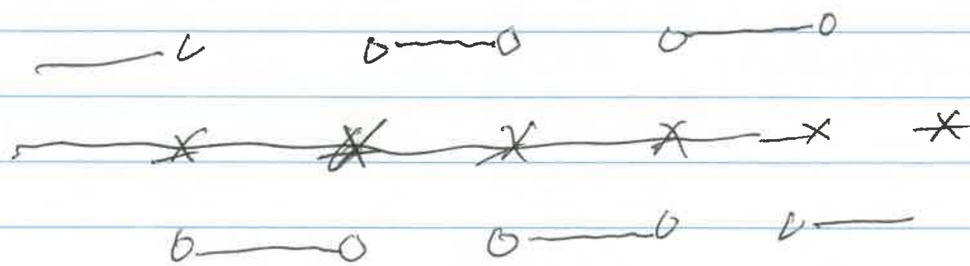
$$f(x) = \begin{cases} -1, & -\pi < x \le 0 \\ 1, & 0 < x \le \pi \end{cases}$$



Periodic ext.



$* \longrightarrow$ FS converges to 0 at x,



Soln. $$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, \underbrace{e^{-inx}\,dx}_{d\left(\frac{e^{-inx}}{-in}\right)} = \begin{cases} 0, & n = 0 \\ \frac{1}{2\pi}\left[\frac{x e^{-inx}}{-in}\Big|_{-\pi}^{\pi} + \cancel{\frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx}^{\,0}\right] \end{cases}$$

$$n \neq 0 \quad = \frac{1}{2\pi}\left[\frac{\pi e^{-in\pi} - (-\pi) e^{in\pi}}{-in}\right]$$

$$\Rightarrow \alpha_n = \frac{1}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi}(-1)^n}{-in}$$

$$\alpha_n = \frac{i(-1)^n}{n}.$$

$$\therefore\ x = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{i(-1)^n}{n} e^{inx}$$

Note: ~~sin/cosine fns~~

$$\cancel{\mathrm{Re}(x) = x = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)}$$

$$\cancel{\Rightarrow x = -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n \sin(nx)}{n}}$$

odd in n.

$$\cancel{x = \sum_{k=1}^{\infty} (-1)}$$

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Parseval's Thm (Complex Form).

$$\int_{-\pi}^{\pi} |x|^2 dx = 2\int_0^{\pi} x^2 dx = \frac{2}{3}\pi^3,$$

$$|\alpha_n|^2 = \left|\frac{i(-1)^n}{n}\right|^2 = \frac{|i|^2 |(-1)^n|^2}{n^2} = \frac{1}{n^2}.$$

$$\frac{2\pi^3}{3} = \left(\sum_{\substack{n \neq 0 \\ n=-\infty}}^{\infty} \frac{1}{n^2}\right) 2\pi.$$

Note: $\displaystyle\sum_{\substack{n \neq 0 \\ n=-\infty}}^{\infty} \frac{1}{n^2} = \sum_{n=-\infty}^{-1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2}$

Now, $\displaystyle\sum_{n=-\infty}^{-1} \frac{1}{n^2} \underset{k=-n}{=} \sum_{k=1}^{\infty} \frac{1}{(-k)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}.$

Go back to n

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\Rightarrow \frac{2\pi^3}{3} = 2\pi\left(\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \cdot 2\right)$$

$$\Rightarrow \frac{2\pi^3}{3} = 4\pi\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$