Throughout these notes, \( \mathcal{H} \) denotes a separable Hilbert space. We will use the notation \( \mathcal{B}(\mathcal{H}) \) to denote the set of bounded linear operators on \( \mathcal{H} \). We also note that \( \mathcal{B}(\mathcal{H}) \) is a Banach space, under the usual operator norm.

1 Compact and Precompact Subsets of \( \mathcal{H} \)

Definition 1.1. A subset \( S \) of \( \mathcal{H} \) is said to be compact if and only if it is closed and every sequence in \( S \) has a convergent subsequence. \( S \) is said to be precompact if its closure is compact.

Proposition 1.2. Here are some important properties of compact sets.

1. Every compact set is bounded.

2. A bounded set \( S \) is precompact if and only if every bounded sequence has a convergent subsequence.

3. Let \( \mathcal{H} \) be finite dimensional. Every closed, bounded subset of \( \mathcal{H} \) is compact.

4. In an infinite dimensional space, closed and bounded is not enough.

Proof. Properties 2 and 3 are left to the reader. For property 1, assume that \( S \) is an unbounded compact set. Since \( S \) is unbounded, we may select a sequence \( \{v_n\}_{n=1}^{\infty} \) from \( S \) such that \( \|v_n\| \to \infty \) as \( n \to \infty \). Since \( S \) is compact, this sequence will have a convergent subsequence, say \( \{v_{n_k}\}_{k=1}^{\infty} \), which still be unbounded. (Why?) Let \( v = \lim_{k \to \infty} v_{n_k} \). Thus, for \( \varepsilon = 1 \) there is a positive integer \( K \) for which \( \|v - v_{n_k}\| < 1 \) for all \( k \geq K \). By the triangle inequality, \( \|v_{n_k}\| \leq \|v\| + 1 \). Now, the right side is bounded, but the left side isn’t, since \( \|v_{n_k}\| \to \infty \) as \( k \to \infty \). This is a contradiction, so \( S \) must be bounded. For property 4, let \( S = \{f \in \mathcal{H} : \|f\| \leq 1\} \). Every o.n. basis \( \{\phi_n\}_{n=1}^{\infty} \) is in \( S \). However, for such a basis \( \|\phi_m - \phi_n\| = \sqrt{2} \), \( n \neq m \). Again, this means there are no Cauchy subsequences in \( \{\phi_n\}_{n=1}^{\infty} \), and consequently, no convergent subsequences. Thus, \( S \) is not compact.
2 Compact Operators

Definition 2.1. Let $K : \mathcal{H} \rightarrow \mathcal{H}$ be linear. $K$ is said to be compact if and only if $K$ maps bounded sets into precompact sets. Equivalently, $K$ is compact if and only if for every bounded sequence $\{v_n\}_{n=1}^\infty$ in $\mathcal{H}$ the sequence $\{Kv_n\}_{n=1}^\infty$ has a convergent subsequence. We denote the set of compact operators on $\mathcal{H}$ by $\mathcal{C}(\mathcal{H})$.

Proposition 2.2. If $K \in \mathcal{C}(\mathcal{H})$, then $K$ is bounded – i.e., $\mathcal{C}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. In addition, $\mathcal{C}(\mathcal{H})$ is a subspace of $\mathcal{B}(\mathcal{H})$.

Proof. We leave this as an exercise for the reader.

We now turn to giving some examples of compact operators. We start with the finite-rank operators. If the range of an operator $K$ is finite dimensional, then we say that $K$ is a finite-rank operator.

Proposition 2.3. Every finite-rank operator $K$ is compact.

Proof. The range of $K$ is finite dimensional, so every bounded subset of the range is compact. Let $S \subseteq \{f \in \mathcal{H} : \|f\| \leq C\}$, where $C$ is fixed. Note that the range of $K$ restricted to $S$ is also bounded: $\|Kf\| \leq \|K\|_{op}\|f\| \leq C\|K\|_{op}$. Thus, $K$ maps a bounded set $S$ into a closed, bounded subset of a finite dimensional subspace of $\mathcal{F}$, which is itself compact. Hence, $K$ is compact.

To describe $K$ explicitly, let $\{\phi_k\}_{k=1}^n$ be a basis for $R(K)$. Then, $Kf = \sum_{k=1}^n a_k \phi_k$. We want to see how the $a_k$’s depend on $f$. Consider $\langle Kf, \phi_j \rangle = \langle f, K^*\phi_j \rangle = \sum_{k=1}^n a_k \langle \phi_k, \phi_j \rangle$. Next let $\psi_j = K^*\phi_j$, so that $\langle f, K^*\phi_j \rangle = \langle f, \psi_j \rangle$. Because $\{\phi_k\}_{k=1}^n$ is a basis, it is linear independent. Hence, the Gram matrix $G_{j,k} = \langle \phi_k, \phi_j \rangle$ is invertible, and so we can solve the system of equations $\langle f, \psi_j \rangle = \sum_{k=1}^n G_{j,k} a_k$. Doing so yields $a_k = \sum_{j=1}^n (G^{-1})_{k,j} \langle f, \psi_j \rangle$. The $a_k$’s are obviously linear in $f$. Of course, a different basis will give a different representation.

Let $\mathcal{H} = L^2[0,1]$. A particularly important set of finite rank operators in $\mathcal{C}(\mathcal{H})$ are ones given by finite rank or degenerate kernels, $k(x, y) = \sum_{k=1}^n \phi_k(x)\overline{\psi_k(y)}$, where the functions involved are in $L^2$. The operator is then $Kf(x) = \int_0^1 k(x, y)f(y)dy$. In the example that we did for resolvents, the kernel was $k(x, y) = xy^2$, and the operator was $Ku(x) = \int_0^1 k(x, y)u(y)dy$. Later, we will show that the Hilbert-Schmidt kernels also yield compact operators. Before, we do so, we will discuss a few more properties of compact operators.
Lemma 2.4. Let \( \{ \phi_n \}_{n=1}^{\infty} \) be an o.n. set in \( \mathcal{H} \) and let \( K \in \mathcal{C}(\mathcal{H}) \). Then, 
\[
\lim_{n \to \infty} K\phi_n = 0.
\]

Proof. Suppose not. Then we may select a subsequence \( \{ \phi_m \} \) of \( \{ \phi_n \}_{n=1}^{\infty} \) for which \( \|K\phi_m\| \geq \alpha > 0 \) for all \( m \). Because \( K \) is compact, we can also select a subsequence \( \{ \phi_k \} \) of \( \{ \phi_m \} \) for which \( \|K\phi_k\| \geq \alpha > 0 \). Now, \( \{ \phi_k \} \) being a subsequence of \( \{ \phi_m \} \) implies that \( \|K\phi_k\| \geq \alpha > 0 \). Taking the limit in this inequality yields \( \|\psi\| \geq \alpha > 0 \). Next, note that 
\[
\lim_{k \to \infty} \langle K\phi_k, \psi \rangle = \|\psi\|^2.
\]
However, 
\[
\lim_{k \to \infty} \langle K\phi_k, \psi \rangle = \lim_{k \to \infty} \langle \phi_k, K^*\psi \rangle = 0,
\]
by Parseval’s theorem. Thus, \( \|\psi\|^2 = 0 \), which is a contradiction.

This lemma is a special case of a more general result. We say that a sequence \( \{ f_n \} \) is weakly convergent to \( f \in \mathcal{H} \) if and only if for all \( g \in \mathcal{H} \) we have 
\[
\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle.
\]
For example, the o.n. set in the lemma weakly converges to 0.

Proposition 2.5. Let \( \{ f_n \} \) weakly converge to \( f \in \mathcal{H} \). If \( K \in \mathcal{C}(\mathcal{H}) \), then 
\[
\lim_{n \to \infty} Kf_n = Kf.
\]
That is, \( K \) maps weakly convergent sequences into “strongly” convergent ones.

Proof. The proof is similar to that of Lemma 2.4. See exercise 4 in assignment 9.

We remark that the converse is true, too. This leads to an alternative characterization of compact operators: \( K \) is compact if and only if \( K \) maps weakly convergent sequences into strongly convergent ones. See the book Functional Analysis, by F. Riesz and B. Sz.-Nagy.

Our next result is one of the most important theorems in the theory of compact operators.

Theorem 2.6. \( \mathcal{C}(\mathcal{H}) \) is a closed subspace of \( \mathcal{B}(\mathcal{H}) \).

Proof. Suppose that \( \{ K_n \}_{n=1}^{\infty} \) is a sequence in \( \mathcal{C}(\mathcal{H}) \) that converges to \( K \in \mathcal{B}(\mathcal{H}) \), in the operator norm. We want to show that \( K \) is compact. Assume the \( \{ v_k \} \) is a bounded sequence in \( \mathcal{H} \), with \( \|v_k\| \leq C \) for all \( k \). Compactness will follow if we can prove that \( K(v_k) \) has a convergent subsequence. The technique for doing this is often called a diagonalization argument. We start with the full sequence and form \( \{ K_1v_k \} \). Since \( K_1 \) is compact, we can select a subsequence \( \{ v_k^{(1)} \} \) such that \( \{ K_1v_k^{(1)} \} \) is convergent. We may carry out the same procedure with \( \{ K_2v_k^{(1)} \} \), selecting a subsequence of \( \{ K_2v_k^{(1)} \} \) that is convergent. Call it \( \{ v_k^{(2)} \} \). Since this is a subsequence of \( \{ v_k^{(1)} \} \), \( \{ K_1v_k^{(2)} \} \) is convergent. Continuing in this way, we construct subsequences.
\{v_k^{(j)}\} for which \{K_m v_k^{(j)}\} is convergent for all \(1 \leq m \leq j\). Next, we let \(\{u_j := v_j^{(j)}\}\), the “diagonal” sequence. This is a subsequence of all of the \(\{v_k^{(j)}\}\)'s. Consequently, for \(n\) fixed, \(\{K_n u_j\}_{j=1}^\infty\) will be convergent. To finish up, we will use an “up, over, and around” argument. Note that for all \(\ell, m,\)

\[
\|Ku_\ell - K u_m\| \leq \|Ku_\ell - K u_\ell\| + \|K u_\ell - K u_m\| + \|K u_m - K u_m\|
\]

Since \(\|Ku_\ell - K u_\ell\| \leq \|K - K\|_{op} \|u_\ell\| \leq 2C\|K - K\|_{op}\) and, similarly, \(\|K u_m - K u_m\| \leq 2C\|K - K\|_{op}\), we have \(\|Ku_\ell - K u_m\| \leq 4C\|K - K\|_{op} + \|K u_\ell - K u_m\|\). Let \(\varepsilon > 0\). First choose \(N\) such that for \(n \geq N\), \(\|K - K\|_{op} < \varepsilon/(8C)\). Fix \(n\). Because \(\{K_n u_\ell\}\) is convergent, it is Cauchy. Choose \(N'\) so large that \(\|K_n u_\ell - K_n u_m\| < \varepsilon/2\) for all \(\ell, m \geq N'\). Putting these two together yields \(\|Ku_\ell - K u_\ell\| \leq \varepsilon\), provided \(\ell, m \geq N'\). Thus \(\{K u_\ell\}\) is Cauchy and therefore convergent.

\[\square\]

**Corollary 2.7.** Hilbert-Schmidt operators are compact.

**Proof.** Let \(H = L^2[0, 1]\) and suppose \(k(x, y) \in L^2(R), R = [0, 1] \times [0, 1]\). The associated Hilbert-Schmidt operator is \(K u = \int_0^1 k(x, y) u(y) dy\). Let \(\{\phi_n\}_{n=1}^\infty\) be an o.n. basis for \(L^2[0, 1]\). With a little work, one can show that \(\{\phi_n(x)\phi_m(y)\}_{n,m=1}^\infty\) is an o.n. basis for \(L^2(R)\). Also, from example 2 in the notes on [Bounded Operators & Closed Subspaces](#), we have that

\[
\|K\|_{op} \leq \|k\|_{L^2(R)}.
\]

Expand \(k(x, y)\) in the o.n. basis \(\{\phi_n(x)\phi_m(y)\}_{n,m=1}^\infty\):

\[
k(x, y) = \sum_{n,m=1}^\infty \alpha_{m,n} \phi_n(x)\phi_m(y), \quad \alpha_{m,n} = \langle k(x, y), \phi_n(x)\phi_m(y) \rangle_{L^2(R)}
\]

Next, let \(k_N(x, y) = \sum_{n,m=1}^N \alpha_{m,n} \phi_n(x)\phi_m(y)\) and also \(K_N\) be the finite rank operator \(K_N u(x) = \int_0^1 k_N(x, y) u(y) dy\). By Parseval’s theorem, we have that \(\|k - K_N\|_{L^2(R)}^2 = \sum_{n,m=N+1}^\infty |\alpha_{m,n}|^2\) and by example 2 mentioned above, \(\|K - K_N\|_{op}^2 \leq \|k - K_N\|_{L^2(R)}^2\), so

\[
\|K - K_N\|_{op}^2 \leq \sum_{n,m=N+1}^\infty |\alpha_{m,n}|^2
\]

Because the series on the right above converges to 0 as \(N \to \infty\), we have \(\lim_{N \to \infty} \|K - K_N\| = 0\). Thus \(K\) is the limit in \(B(L^2[0, 1])\) of finite rank operators, which are compact. By the theorem above, \(K\) is also compact. \[\square\]

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\[1\)See Keener, Theorem 3.5
We now turn to some of the algebraic properties of \( \mathcal{C}(\mathcal{H}) \).

**Proposition 2.8.** Let \( K \in \mathcal{C}(\mathcal{H}) \) and let \( L \in \mathcal{B}(\mathcal{H}) \). Then both \( KL \) and \( LK \) are in \( \mathcal{C}(\mathcal{H}) \).

**Proof.** Let \( \{v_k\} \) be a bounded sequence in \( \mathcal{H} \). Since \( L \) is bounded, the sequence \( \{Lv_k\} \) is also bounded. Because \( K \) is compact, we may find a subsequence of \( \{KLv_k\} \) that is convergent, so \( KL \in \mathcal{C}(\mathcal{H}) \). Next, again assuming \( \{v_k\} \) is a bounded sequence in \( \mathcal{H} \), we may extract a convergent subsequence from \( \{Kv_k\} \), which, with a slight abuse of notation, we will denote by \( \{Kv_j\} \). Because \( L \) is bounded, it is also continuous. Thus \( \{LKv_j\} \) is convergent. It follows that \( LK \) is compact.

**Proposition 2.9.** \( K \) is compact if and only if \( K^* \) is compact.

**Proof.** Because \( K \) is compact, it is bounded and so is its adjoint \( K^* \), in fact \( \|K^*\|_{op} = \|K\|_{op} \). By Proposition 2.8, we thus have that \( KK^* \) is compact. It follows that if \( \{u_n\} \) be a bounded sequence in \( \mathcal{H} \), then we may extract a subsequence \( \{u_j\} \) such that the sequence \( \{KK^*v_j\} \) is convergent. This of course means that this sequence is also Cauchy. Note that

\[
\langle KK^*(v_j - v_k), v_j - v_k \rangle = \langle K^*(v_j - v_k), K^*(v_j - v_k) \rangle = \|K^*(v_j - v_k)\|^2.
\]

From and the fact that \( \{v_j\} \) is bounded, we see that

\[
\|v_j - v_k\| \|KK^*(v_j - v_k)\| \leq \|v_j - v_k\|^2 \|KK^*(v_j - v_k)\| \leq C\|KK^*(v_j - v_k)\|.
\]

Thus,

\[
\|K^*(v_j - v_k)\|^2 \leq C\|KK^*(v_j - v_k)\|
\]

Since \( \{KK^*v_j\} \) is Cauchy, for every \( \varepsilon > 0 \), we can find \( N \) such that whenever \( j, k \geq N \), \( \|KK^*(v_j - v_k)\| < \varepsilon^2/C \). It follows that \( \|K^*(v_j - v_k)\| < \varepsilon \), if \( j, k \geq N \). This implies that \( \{K^*v_j\} \) is Cauchy and therefore convergent.

We want to put this in more algebraic language. Taking \( L \) to be compact in Proposition 2.8, we have that the product of two compact operators is compact. Since \( \mathcal{C}(\mathcal{H}) \) is already a subspace, this implies that it is an algebra. Moreover, by taking \( L \) to be just a bounded operator, we have that \( \mathcal{C}(\mathcal{H}) \) is a two-sided ideal in the algebra \( \mathcal{B}(\mathcal{H}) \). Since \( K \) being compact implies \( K^* \) is compact, \( \mathcal{C}(\mathcal{H}) \) is closed under the operation of taking adjoints; thus, \( \mathcal{C}(\mathcal{H}) \) is a *-ideal. Finally, including the result of Theorem 2.6, we have that \( \mathcal{C}(\mathcal{H}) \) is a closed under limits. We summarize these results as follows.

**Theorem 2.10.** \( \mathcal{C}(\mathcal{H}) \) is a closed, two-sided, *-ideal in \( \mathcal{B}(\mathcal{H}) \).
We remark that a closed *-algebra in $\mathcal{B}(\mathcal{H})$ is called a C*-algebra. So, $C(\mathcal{H})$ is a C*-algebra that is also a two-sided ideal in $\mathcal{B}(\mathcal{H})$. 

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Next: the closed range theorem