X-ray Tomography & Integral Equations
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X-ray Tomography. An important part of X-ray tomography – the CAT scan – is solving a mathematical problem that goes back to the earlier twentieth century work of the mathematician Johann Radon: Suppose that there is a function \( f(x, y) \) defined in a region of the plane and that all we know about \( f \) is the collection of line integrals \( \int_L f(x(s), y(s))ds \) over each line \( L \) that intersects the region. (See Figure 1.) The problem is to find \( f \), given this information.

Figure 1: The region where \( f \) is defined and a typical line \( L \) cutting the region are shown. \( L \) is specified by \( \rho \) and the angle \( \theta \).

We will assume that the region where \( f \) is defined is a disk \( D := \{ |x| \leq 1 \} \). In Figure 1 the function is shown as having compact support in \( D \). The unit vector \( n \) that is normal to \( L \) and points away from the origin is \( n = \cos(\theta)i + \sin(\theta)j \). The tangent pointing upward is \( t = -\sin(\theta)i + \cos(\theta)j \).

\(^1\)This is an attenuation coefficient in a CAT scan.
If we let \( s \geq 0 \) be the arc length starting at the point \( \rho \mathbf{n} \), then any point \( \mathbf{x} \) above \( \rho \mathbf{n} \) is specified by \( \mathbf{x} = s \mathbf{t} + \rho \mathbf{n} \). If \( \mathbf{x} \) is below \( \rho \mathbf{n} \), then it is specified by \( \mathbf{x} = -s \mathbf{t} + \rho \mathbf{n} \).

We will work with \( \mathbf{x} \) above the vector \( \rho \mathbf{n} \). Express \( \mathbf{x} \) in terms of polar coordinates \((r, \phi)\), \( \mathbf{x} = r \cos(\phi) \mathbf{i} + r \sin(\phi) \mathbf{j} \). Of course, \( r = |\mathbf{x}| \). Comparing this with \( \mathbf{x} = s \mathbf{t} + \rho \mathbf{n} \), we see that \( r^2 = s^2 + \rho^2 \) and \( \rho = \mathbf{x} \cdot \mathbf{n} = r \cos(\phi - \theta) \).

Since \( \mathbf{x} \) is above \( \rho \mathbf{n} \), we have that \( \phi \geq \theta \) and thus \( \phi = \theta + \cos^{-1} \left( \frac{\rho}{r} \right) \).

When \( \mathbf{x} \) is below \( \rho \mathbf{n} \), \( \phi \leq \theta \) and \( \phi = \theta - \cos^{-1} \left( \frac{\rho}{r} \right) \).

Breaking the integral \( \int_L f(\mathbf{x}(s))ds \) into two pieces, making the change of variables \( s = \sqrt{r^2 - \rho^2} \), \( ds = \frac{(r^2 - \rho^2)^{-1/2}}{r}dr \), and noting that \( \rho \leq r \leq 1 \), we have

\[
\int_L f(\mathbf{x}(s))ds = 2 \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{\rho}^{1} \hat{f}_n(r) \frac{\cos(n \cos^{-1}(\rho/r))}{\sqrt{r^2 - \rho^2}} rdr.
\]

Assuming the \( f(\mathbf{x}) = f(r, \phi) \) is smooth enough, we can expand it in a Fourier series in \( \phi \),

\[
f(r, \phi) = \sum_{n=-\infty}^{\infty} \hat{f}_n(r) e^{in\phi},
\]

and then replace \( f \) in the integral on the right above by this series. Again making the assumption that interchanging sum and integral is possible and manipulating the resulting expression, we have

\[
\int_L f(\mathbf{x}(s))ds = 2 \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{\rho}^{1} \hat{f}_n(r) \frac{\cos(n \cos^{-1}(\rho/r))}{\sqrt{r^2 - \rho^2}} rdr.
\] (1)

Since the line \( L \) is specified by the angle \( \theta \) and distance \( \rho \), the integral over \( L \) is a function of \( \theta \) and \( \rho \), which we denote by \( F(\rho, \theta) \). In addition, the expression \( T_n(\rho/r) := \cos(n \cos^{-1}(\rho/r)) \) is actually an \( n \)th degree Chebyshev polynomial. For example, \( T_2(\rho/r) = 2 \cos^2(\cos^{-1}(\rho/r)) - 1 = 2(\rho/r)^2 - 1 \). Using these two facts in connection with (1) we have

\[
F(\rho, \theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{\rho}^{1} \hat{f}_n(r) \frac{T_n(\rho/r)r}{\sqrt{r^2 - \rho^2}} dr.
\] (2)
The Fourier series for \( F(\rho, \theta) = \sum_{n=-\infty}^{\infty} \hat{F}_n(\rho)e^{in\theta} \). Comparing it with the series in (2) we arrive at

\[
\hat{F}_n(\rho) = \int_{\rho}^{1} \hat{f}_n(r) \frac{T_n(\rho/r)r}{\sqrt{r^2 - \rho^2}} dr, \quad n \in \mathbb{Z}.
\] (3)

The point is that \( F(\rho, \theta) = \int f(x(s))ds \) is known, and so the Fourier coefficients \( \hat{F}_n(\rho) \) are all known. The problem of finding \( f \), given \( F \), is thus equivalent to solving the integral equations in (3) for the \( \hat{f}_n(r)'s \) and recovering \( f(r, \phi) \) from its Fourier series. In fact, these integral equations have exact solutions (see Keener, §3.7):

\[
\hat{f}_n(r) = -\frac{1}{\pi} \frac{d}{dr} \int_{\rho}^{1} rT_n(\rho/r) \hat{F}_n(\rho) \rho \sqrt{\rho^2 - r^2} d\rho, \quad n \in \mathbb{Z}.
\] (4)

Classification of integral equations. Certain types of integral equations come up often enough that they are grouped into classes, which are described below. There, the function \( f \) and kernel \( k(x, y) \) are known, \( u \) is the unknown function to be solved for, and \( \lambda \) is a parameter. The integral equations in (3) are Volterra equations of the first kind.

**Fredholm Equations**

1\(^{st}\) kind. \( f(x) = \int_{a}^{b} k(x, y)u(y)dy \).

2\(^{nd}\) kind. \( u(x) = f(x) + \lambda \int_{a}^{b} k(x, y)u(y)dy \).

**Volterra Equations**

1\(^{st}\) kind. \( f(x) = \int_{a}^{x} k(x, y)u(y)dy \).

2\(^{nd}\) kind. \( u(x) = f(x) + \lambda \int_{a}^{x} k(x, y)u(y)dy \).

Acknowledgments Figure 1 is from the article “A small note on Matlab iradon and the all-at-once vs. the one-at-a-time method,” by Nasser M. Abbasi. July 17, 2008. The figure was downloaded on November 10, 2013, from the website

http://12000.org/my_notes/note_on_radon/

note_on_radon/note_on_radon.htm

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