Error Analysis for the BCR Quadrature Formula

We will assume that $f(x)$ is twice continuously differentiable, and that its second derivative is uniformly bounded. Our object is to estimate the error made in using the BCR quadrature formula to obtain the integrals

$$f_j[n] := \int_{-\infty}^{\infty} f(x) 2^{-j/2} \varphi(2^{-j}x - n)dx.$$

A simple change of variables puts the integral above into the form

$$f_j[n] = \int_{-\infty}^{\infty} f(x + 2^j n) 2^{-j/2} \varphi(2^{-j}x)dx.$$

Using the fact that $\varphi(x) = 0$ for all x outside of the interval $[0, 3]$, we arrive at

$$f_j[n] = \int_{0}^{3\cdot 2^j} f(x + 2^j n) 2^{-j/2} \varphi(2^{-j}x)dx$$

According to the BCR formula,

$$f_j[n] \approx 2^{j/2} \left(q_0 f(n \cdot 2^j) + q_1 f((n+1)2^j) \right), \tag{1}$$

with the formula being exact for functions of the form $a + bx$.

We begin our error analysis by first expanding $f(x)$ about $x = n \cdot 2^j$ via Taylor's Theorem with Remainder, we have that

$$f(x) = \underbrace{f(n \cdot 2^j) + f'(k \cdot 2^j)(x - n \cdot 2^j)}_{F(x)} + f''(c)(x - n \cdot 2^j)^2/2, \tag{2}$$

where c is between x and $k \cdot 2^j$. Replacing f in the integrals for $f_j[n]$ by this formula yields

$$f_j[n] = \int_{0}^{3\cdot 2^j} F(x + n \cdot 2^j) 2^{-j/2} \varphi(2^{-j}x)dx + \underbrace{\int_{0}^{3\cdot 2^{-j}} \left(f''(c)x^2/2 \right) 2^{-j/2} \varphi(2^{-j}x)dx}_{E_1} \tag{3}$$

Since $F(x)$ is linear, the first integral can be computed exactly via the BCR formula. Thus our last equation becomes

$$f_j[n] = 2^{j/2} \Bigg(q_0 F(n \cdot 2^j) + q_1 F((n+1)2^j)) \Bigg) + E_1. \tag{4}$$

From Taylor's formula (2), we see that $F(x) = f(x) - f''(c)(x - n\cdot 2^j)^2/2$. Consequently, we have that

$$F(n \cdot 2^j) = f(k \cdot 2^j) \quad \text{and} \quad F\big((n+1)2^j\big) = f\big((n+1)2^j\big) - 2^{2j}\big(f''(c)/2\big).$$

Inserting these in (4), we arrive at

$$\begin{aligned}
f_j[n] &= 2^{j/2}\left(q_0 f(n\cdot 2^j) + q_1 f((n+1)2^j)) - 2^{2j}\big(q_1 f''(c)/2\big)\right) + E_1 \\
&= \underbrace{2^{j/2}\left(q_0 f(n\cdot 2^j) + q_1 f((n+1)2^j))\right)}_{BCR} + \underbrace{E_1 - 2^{5j/2}\big(q_1 f''(c)/2\big)}_{E} \qquad (5)
\end{aligned}$$

The total error made by replacing the actual integral with the BCR quadrature formula is thus $E = E_1 - 2^{5j/2}\big(q_1 f''(c)/2\big)$. Since $f''(x)$ is assumed to be uniformly bounded—*i.e.*, there is a constant C_0 such that $|f''(x)| \le C_0$ for all x—, it is clear that

$$|E| \le |E_1| + \Big(|q_1|C_0/2\Big)2^{5j/2}. \qquad (6)$$

To estimate E_1, we again use $|f''(x)| \le C_0$. From (3), we have

$$\begin{aligned}
|E_1| \le &\int_0^{3\cdot 2^j} |f''(c)/2|\, 2^{-j/2} x^2 |\varphi(2^{-j}x)| dx \\
&\le (C_0/2) 2^{-j/2} \int_0^{3\cdot 2^j} x^2 |\varphi(2^{-j}x)| dx \\
&\le 2^{5j/2}(C_0/2)\int_0^3 u^2|\varphi(u)|du \quad (u = 2^{-j}x) \\
&\le 2^{5j/2}(C_0/2)\left(\int_0^3 u^4 du\right)^{1/2}\left(\int_0^3 \varphi^2(u)du\right)^{1/2} \quad \text{(Schwarz's inequality.)} \\
&\le 2^{5j/2}(C_0/2)(3^5/5)^{1/2}\|\varphi\| \\
&\le 3.49 \times C_0 2^{5j/2} \quad (\|\varphi\| = 1). \qquad (7)
\end{aligned}$$

Using the value $q_1 = 0.3665$ and combining (6) and (7), we arrive at this estimate for the total error, E:

$$|E| \le (3.7 \times C_0)2^{5j/2}. \qquad (8)$$