Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 13, 2021

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = x^2\phi(x)$ for some $\phi \in \mathcal{D}$ if and only if $\psi(0) = 0$ and $\psi'(0) = 0$.
(d) Use 2(c) to find all $T \in \mathcal{D}'$ that satisfy $x^2T(x) = 0$.

Problem 2. Let $\mathcal{P}$ be the set of all polynomials.

(a) State and sketch a proof of the Weierstrass approximation theorem.
(b) let $\mathcal{H} = L^2_w[0,1]$, where the inner product is $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}w(x)dx$ and where $w \in C[0,1]$, $w(x) \geq c > 0$ on $[0,1]$. Show that $\mathcal{P}$ is dense in $L^2_w[0,1]$. (You may use the density of $C[0,1]$ in $L^2[0,1]$.)
(c) Let $\mathcal{U} := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that $\mathcal{U}$ is a complete set in $L^2_w[0,1]$.

Problem 3. Suppose that K is a compact operator and $Tu(x) := \int_{-\infty}^{\infty} e^{-|x-y|^2}u(y)dy$.

(a) Show that if $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal set, then $\lim_{j\to\infty} K\phi_j = 0$.
(b) Show that T is a bounded operator on $L^2(\mathbb{R})$.
(c) The set $\phi_j = \chi_{[j,j+1]}$ is an orthonormal basis for $L^2(\mathbb{R})$. Use translation invariance to show that $\|T\phi_j\| = \|T\phi_0\|$.
(d) Show that T is *not* compact.

Problem 4. Let $Lu = x^2u'' + 2xu' - 2u$. $D_L = \{u \in L^2[1,2]],\ u'(1) = 0, u(2) = 0\}$. You are given that x, x^{-2} are homogenous solutions.

(a) Show that $L = L^*$.
(b) Find the Green's function G. Show that $Ku(x) = \int_1^2 G(x,y)u(y)dy$ is compact and selfadjoint.
(c) State the spectral theorem for compact, self-adjoint operators. Use it to show that the (normalized) eigenfunctions of the eigenvalue problem $Lu + \lambda u = 0$, $u \in D_L$, form a complete orthonormal set in $L^2[1,2]$.