Applied Analysis Part

January 10, 2024

Name: ____

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let $\mathcal{P}$ be the set of all polynomials.

(a) State and sketch a proof of the Weierstrass Approximation Theorem.[1]
(b) Use (a) to show that $\mathcal{P}$ is dense in $L^2[0,1]$. (You may use the the fact that $C[0,1]$ is dense in $L^2[0,1]$.)
(c) Let $U := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that U is a complete set in $L^2[0,1]$.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Show that $\psi \in \mathcal{D}$ satisfies $\psi = \phi''$ for some $\phi \in \mathcal{D}$ if and only if
$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \text{ and } \int_{-\infty}^{\infty} x\psi(x)dx = 0.$$
(c) Find all distributions $T \in \mathcal{D}'$ such that $T''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1)$.

Problem 3. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Fredholm Alternative.
(b) State the Closed Range Theorem.
(c) Let $\mathcal{H} = L^2[0,1]$. Define the kernel $k(x,y) := x^3y^2$ and let $Ku(x) = \int_0^1 k(x,y)u(y)dy$. Show that K is in $\mathcal{C}(\mathcal{H})$.
(d) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (c) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0,1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Consider the kernel $k(x,y) = \sum_{n=0}^{\infty}(1+n)^{-2}P_{n+1}(x)P_n(y)$, where the P_n's are the orthogonal set of Legendre polynomials, relative to $L^2[-1,1]$. They are normalized so that $\int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}$.

(a) Show that $Ku(x) = \int_{-1}^{1} k(x,y)u(y)dy$ is a compact operator on $L^2[-1,1]$.
(b) Determine the spectrum of K.

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[1] You may use these identities involving the Bernstein polynomials. The last two identities start the sum at $j = 0$, rather than $j = 1$.
$$1 = \sum_{j=0}^{n} \beta_{j,n}(x), \quad x = \sum_{j=0}^{n} \frac{j}{n}\beta_{j,n}(x) \qquad \frac{1}{n}x + (1 - \frac{1}{n})x^2 = \sum_{j=0}^{n} \frac{j^2}{n^2}\beta_{j,n}(x).$$