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ORIENTABILITY OF REAL TORIC VARIETIES

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Abstract. We characterize the orientability of an abstract real toric variety as well as the orientability of a toric subvariety of a sphere. We also determine the number of components of the smooth locus of a toric variety. These results are proven for an extension of the Davis-Januskiewicz notion of a small cover to singular spaces. We characterize the orientability of toric varieties associated to posets, and discuss an application to the study of real solutions to systems of polynomial equations.

Introduction

Real toric varieties appear in many applications of mathematics [1, 5, 7] and are interesting objects in their own right [3]. Davis and Januszkiewicz [2] introduced the notion of a small cover of a simple convex polytope as a generalization of smooth projective real toric varieties. Nakayama and Nishimura [6] characterized when a small cover is orientable. The orientability of a projective real toric variety and of its double cover in a sphere plays a key role in our study of lower bounds for systems of polynomial equations [8] in which we gave criteria that imply, but do not characterize, this orientability.

We extend those results [6, 8], characterizing when the smooth points of a toric variety are orientable. This is sufficient for our applications to polynomial equations as toric varieties are smooth in codimension one. Our results apply not necessarily projective toric varieties and to a generalization of Davis and Januszkiewicz's notion of a small cover (which are manifolds) to not necessarily smooth spaces. When a real toric variety is projective, it may be lifted to the sphere which has a two-to-one map to real projective space. We also characterize the orientability of these spherical toric varieties, and determine the number of connected components of the smooth points of these spaces.

We review the construction of real toric varieties and spherical toric varieties in Section 1, where we also define a not necessarily smooth small cover. Our main results are established in Section 2, and we conclude in Section 3 with applications and explain the motivation from the study of real solutions to systems of polynomial equations.

1. Constructions of real toric varieties

Real toric varieties, singular small covers, and toric subvarieties of the sphere are obtained by gluing the real torus $\mathbb{T}^n := (\mathbb{R}^\times)^n$ or $\{\pm 1\} \times \mathbb{T}^n = \mathbb{T}^{n+1}/\mathbb{R}_\geq$ along copies of $\mathbb{T}^{n-1}$, one copy for each vector in a collection of integer vectors. There are further gluings and identifications in higher codimension, which presents these spaces as explicit cell complexes. They are smooth at the points of their dense torus $\mathbb{T}^n$ (or at $\{\pm 1\} \times \mathbb{T}^n$) and the attached tori $\mathbb{T}^{n-1}$, and so their orientability is determined by the gluing along the tori $\mathbb{T}^{n-1}$. We describe this gluing construction for these different types of spaces.

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Complex toric varieties are normal varieties over $\mathbb{C}$ equipped with an action of an algebraic torus $(\mathbb{C}^\times)^n$ having a dense orbit. They are classified by rational fans $\Sigma \subset \mathbb{R}^n$, which encode their construction as a union of affine toric varieties U_σ, one for each cone $\sigma \in \Sigma$. A toric variety may also be viewed as a disjoint union of torus orbits $\mathcal{O}_\sigma$, one for each cone $\sigma \in \Sigma$, with the dimension of $\mathcal{O}_\sigma$ equal to the codimension of the cone σ. The dense orbit $\mathcal{O}_0$ coincides with the smallest affine patch U_0, and both are associated to the smallest cone in the fan, the origin 0. See [4] for a complete description.

A toric variety has a canonical set Y of real points which are obtained from the real points of the sets U_σ and orbits $\mathcal{O}_\sigma$ of the construction. This is described in [4, Ch. 4]. The dense orbit $\mathcal{O}_0(\mathbb{R}) \simeq \mathbb{T}^n$ is isomorphic to $(\mathbb{R}^\times)^n = (\mathbb{R}_{>0})^n \times \{\pm 1\}^n$, which has 2^n components, each a topological n-ball. The subgroup $\{\pm 1\}^n \subset \mathbb{T}^n$ acts on the real toric variety Y, permuting the components of $\mathcal{O}_0(\mathbb{R})$. The orbit space of Y under the group $\{\pm 1\}^n$ is isomorphic to the closure $Y_\geq$ of any component of $\mathcal{O}_0(\mathbb{R})$ in the usual topology (not Zariski!) on Y. Each orbit $\mathcal{O}_\sigma(\mathbb{R})$ has a unique component meeting (and in fact contained in) $Y_\geq$. We call this component F_σ a *face* of $Y_\geq$, which is isomorphic to $(\mathbb{R}_{>0})^{n-\dim(\sigma)}$. Those faces endow $Y_\geq$ with the structure of a cell complex that is dual to the fan Σ. That is, the intersection of the closures $\overline{F_\sigma}$, $\overline{F_\tau}$ of two faces is the closure $\overline{F_\rho}$ of a face where ρ is the minimal cone of Σ containing both σ and τ, or the intersection is empty if there is no cone containing both σ and τ.

The integer points in a cone σ of Σ form a subsemigroup of $\mathbb{Z}^n$ whose image in $(\mathbb{Z}/2\mathbb{Z})^n = \{\pm 1\}^n$ is a subgroup $\overline{\sigma}$ of $\{\pm 1\}^n$. This subgroup $\overline{\sigma}$ is the isotropy subgroup of the face F_σ of $Y_\geq$. This gives the following description of Y as a quotient space of $Y_\geq \times \{\pm 1\}^n$.

Proposition 1.1. *The real toric variety Y is obtained as the quotient of the cell complex $Y_\geq \times \{\pm 1\}^n$ by the equivalence relation where*

$$(p,\xi) \sim (q,\eta) \iff p = q \text{ and } \xi\overline{\sigma} = \eta\overline{\sigma}\,, \text{ where } p \text{ lies in the face } F_\sigma.$$

A *facet* of $Y_\geq$ is a face F_σ corresponding to a one-dimensional cone σ. The toric variety Y is smooth at points corresponding to facets, but may not be smooth at points of lower dimensional faces. If we let $Y^\circ_\geq$ be the union of the dense face F_0 and its facets, then

$$Y^\circ := (Y^\circ_\geq \times \{\pm 1\}^n)/\sim$$

consists of smooth points of Y. We study the orientability of Y° (equivalent to that of Y) and its number of connected components.

We generalize this construction. Let P be a finite ranked poset with minimal element 0 and rank at most n where two elements $\sigma, \tau \in P$ either have a unique upper bound in P or else have no upper bound in P. The cones σ in a rational fan in $\mathbb{R}^n$ form such a poset. We also require a cell complex Δ and a system $\mathcal{S}$ of subgroups of $\{\pm 1\}^n$ indexed by P. More specifically, suppose that we have a collection $\mathcal{S} := \{\overline{\sigma} \mid \sigma \in P\}$ of subgroups of $\{\pm 1\}^n$ where $\overline{\sigma} \simeq \{\pm 1\}^{\mathrm{rank}(\sigma)}$, and if $\sigma \subset \tau$, then $\overline{\sigma} \subset \overline{\tau}$. Also suppose that we have a cell complex Δ with cells (called faces) indexed by elements of P,

$$\Delta = \coprod_{\sigma \in P} F_\sigma,$$

where each face F_σ is a cell of dimension $n - \mathrm{rank}(\sigma)$, which for concreteness, we identify with the interior of the closed unit ball in $\mathbb{R}^{n-\mathrm{rank}(\sigma)}$. We further suppose that:

- Δ is a subset of the closed ball $\overline{F_0}$ in $\mathbb{R}^n$,

- the closure of a face F_σ in $\mathbb{R}^n$ is homeomorphic to the closed ball of dimension $n-\operatorname{rank}(\sigma)$, and
- given $\sigma, \tau \in P$, the closures of the faces F_σ and F_τ either do not meet (if σ and τ have no upper bound in P), or their intersection is the closure of the face F_ρ, where ρ is the least upper bound of σ and τ in P.

Definition 1.2. Given a cell complex Δ, ranked poset P and system $\mathcal{S}$ of subgroups of $\{\pm 1\}^n$ as above, the *small cover $Y(\Delta, \mathcal{S})$* of Δ is the quotient

$$(\Delta \times \{\pm 1\}^n)/\sim\,,$$

where $(p, \xi) \sim (q, \eta)$ if and only if $p = q$ and $\xi\overline{\sigma} = \eta\overline{\sigma}$, where p lies in the face F_σ.

Observe that $Y(\Delta, \mathcal{S})$ is equipped with a natural action of $\{\pm 1\}^n$ whose orbit space is Δ, where the orbit of a face F_σ is identified with $F_\sigma \times \{\pm 1\}^n/\overline{\sigma} \simeq \mathbb{T}^{n-\operatorname{rank}(\sigma)}$. In particular, it is a $\{\pm 1\}^n$-equivariant compactification of $\mathbb{T}^n$.

A toric variety Y associated to a fan Σ is a small cover where P is the set of cones in the fan, $\Delta = Y_{\geq}$, and $\mathcal{S} = \{\overline{\sigma} \mid \sigma \in \Sigma\}$.

The points of $Y(\Delta, \mathcal{S})$ corresponding to the big cell F_0 and to facets F_σ are points where $Y(\Delta, \mathcal{S})$ is a topological manifold. Write Δ° for the union of the big cell and the facets, and $Y^\circ(\Delta, \mathcal{S}) = (\Delta^\circ \times \{\pm 1\}^n)/\sim$ for this subset of the smooth points of $Y(\Delta, \mathcal{S})$.

Projective toric varieties admit a second construction. Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional polytope with integer vertices, and normal fan $\Sigma \subset \mathbb{R}^n$. Then the toric variety Y_Σ associated to Σ has a projective embedding given by Δ as follows. Assume that the integer points $\Delta \cap \mathbb{Z}^n$ generate $\mathbb{Z}^n$. Let $\mathbb{P}^\Delta$ be the projective space with coordinates indexed by $\Delta \cap \mathbb{Z}^n$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ the monomial with exponent α. Then we have an injection

$$\varphi_\Delta \,:\, \mathbb{T}^n \;\ni\; y \;\longmapsto\; [y^\alpha \mid \alpha \in \Delta \cap \mathbb{Z}^n]\,, \tag{1.1}$$

where $[\cdots]$ denotes homogeneous coordinates for $\mathbb{P}^\Delta$, where we identify points that are proportional. The closure Y_Δ of the image of this map is isomorphic to the toric variety Y_Σ, and the cell complex $Y^\circ_{\geq}$ is identified with the polytope Δ.

The unit sphere $\mathbb{S}^\Delta \subset \mathbb{R}^\Delta$ has a two-to-one map to the projective space $\mathbb{P}^\Delta$, and we define Y^+_Δ to be the pullback of Y_Δ along this map. The sphere $\mathbb{S}^\Delta$ has homogeneous coordinates $(x_\alpha \mid \alpha \in \Delta \cap \mathbb{Z}^n)$, where we identify points that are proportional with a positive constant of proportionality. The group $\{\pm 1\}^{n+1}$ acts on $\mathbb{S}^\Delta$ where the initial (0th) coordinate acts as the antipodal map—global multiplication by ± 1 and the remaining coordinates $\{\pm 1\}^n$ act through the map φ_Δ (1.1),

$$(g_0, g).(x_\alpha \mid \alpha \in \Delta \cap \mathbb{Z}^n) \;=\; (g_0 g^\alpha x_\alpha \mid \alpha \in \Delta \cap \mathbb{Z}^n)\,.$$

The faces of Y^+_Δ are its intersections with coordinate subspaces $\mathbb{S}^F$ of $\mathbb{S}^\Delta$ corresponding to faces F of Δ,

$$\mathbb{S}^F \;:=\; \{(x_\alpha \mid \alpha \in \Delta \cap \mathbb{Z}^n) \mid x_\alpha = 0 \text{ if } \alpha \notin F \cap \mathbb{Z}^n\}\,.$$

The isotropy subgroup of $\mathbb{S}^F$ is

$$\{(g_0, g) \mid g_0 g^\alpha = 1 \text{ for } \alpha \in F \cap \mathbb{Z}^n\}\,.$$

Vectors b in the normal cone σ_F to a face F of Δ have constant dot product with elements of F—define $b \cdot F$ to be this constant. Then the subgroup

$$\overline{\sigma}_F{}^+ \;:=\; \{(-1)^{(b \cdot F, b)} \mid b \in \sigma_F\} \;\subset\; \{\pm 1\}^{n+1}$$

is the isotropy group of $\mathbb{S}^F$, and therefore of the corresponding face of Y^+_Δ.

Proposition 1.3. *The spherical toric variety* Y^+_Δ *is obtained as the quotient of the cell complex* $\Delta \times \{\pm 1\}^{n+1}$ *by the equivalence relation*

$$(p,\xi) \sim (q,\eta) \iff p = q \quad \text{and} \quad \xi\overline{\sigma}_F{}^+ = \eta\overline{\sigma}_F{}^+\,, \quad \text{where } p \text{ lies in the face } F.$$

2. Proofs

We characterize the orientability of a general (not necessarily smooth) small cover, and determine its number of components. Our elementary methods follow those of Nakayama and Nishimura [6]. We also establish the analogous results for spherical toric varieties Y^+_Δ.

Theorem 2.1. *Let* $Y(\Delta, S)$ *be a small cover of dimension* n.

1. $Y^\circ(\Delta, S)$ *is orientable if and only if there exists a basis of* $\{\pm 1\}^n$ *such that each* $\overline{\sigma} \in S$ *for* $\sigma \in P$ *of rank* 1 *is generated by a product of an odd number of basis vectors.*
2. *The components of* $Y^\circ(\Delta, S)$ *are naturally indexed by* $\{\pm 1\}^n / \langle \overline{\sigma} \mid \operatorname{rank}(\sigma) = 1\rangle$.

Thus $Y^\circ(\Delta, S)$ has 2^{n-k} connected components, where k is the rank of the subgroup $\langle \overline{\sigma} \mid \operatorname{rank}(\sigma) = 1\rangle$ of $\{\pm 1\}^n$.

Proof. For each $\sigma \in P$ with rank 1, let g_σ be the generator of $\overline{\sigma} \simeq \mathbb{Z}/2\mathbb{Z}$. Then $Y^\circ := Y^\circ(\Delta, S)$ is obtained by gluing (Δ, ξ) and (Δ, η) along F_σ whenever $\xi = \eta g_\sigma$ for some $\sigma \in P$ of rank 1, so the number of connected components of Y° is equal to the number of orbits of Y° under the action of $\langle \overline{\sigma} \mid \operatorname{rank}(\sigma) = 1\rangle$.

The space Y° is orientable if and only if its top integral homology group $H_n(Y^\circ, \mathbb{Z})$ is notrivial. This group is the kernel $\ker\partial$ of the differential in the cellular chain complex of the cell complex Y°,

$$C_n \xrightarrow{\ \partial\ } C_{n-1}\,.$$

Here C_n is the free abelian group generated by

$$\{\Delta\} \times \{\pm 1\}^n \;=\; \{(\Delta, \xi) \mid \xi \in \{\pm 1\}^n\}$$

and C_{n-1} is the free abelian group generated by

$$\{[F_\sigma, \xi] \mid \sigma \in P, \operatorname{rank}(\sigma) = 1, \xi \in \{\pm 1\}^n\} \,/ \sim\,,$$

where $[F_\sigma, \xi] \sim [F_\sigma, \eta]$ whenever $\xi\overline{\sigma} = \eta\overline{\sigma}$, which is $[F_\sigma, \xi] \sim [F_\sigma, \xi g_\sigma]$. Orient each facet F_σ so that

$$\partial(\Delta) \;=\; \sum_{\operatorname{rank}(\sigma)=1} F_\sigma\,.$$

Consider an n-cycle

$$X \;=\; \sum_{\xi \in \{\pm 1\}^n} n_\xi \cdot (\Delta, \xi) \;\in\; C_n$$

on Y°, where $n_\xi \in \mathbb{Z}$. Then

$$\partial(X) \;=\; \sum_{\xi \in \{\pm 1\}^n} n_\xi \sum_{\operatorname{rank}(\sigma)=1} [F_\sigma, \xi] \;=\; \sum_{\operatorname{rank}(\sigma)=1} \;\sum_{\xi \in \{\pm 1\}^n / \langle g_\sigma \rangle} (n_\xi + n_{\xi g_\sigma})[F_\sigma, \xi]\,.$$

Hence an n-cycle X lies in $\ker\partial$ if and only if $n_\xi = -n_{\xi g_\sigma}$ for all ξ in $\{\pm 1\}^n$ and σ of rank 1. Equivalently, $n_\xi = (-1)^k n_{\xi g_{\sigma_1} \cdots\, g_{\sigma_k}}$ for all $\xi \in \{\pm 1\}^n$ and σ_i of rank 1.

We show that $\ker\partial$ is non-trivial if and and only if there exists a basis $e_1, \ldots, e_n$ of $\{\pm 1\}^n$ such g_σ is a product of an odd number of basis vectors, for each element $\sigma \in P$ of rank one. Let $\mathbb{O}$ be the set of generators g_σ of $\overline{\sigma}$ for rank one elements $\sigma \in P$.

Suppose that there exists a basis $e_1, \dots, e_n$ of $\{\pm 1\}^n$ such that each $g_\sigma \in \mathbb{O}$ is a product of an odd number of basis vectors. For $\xi \in \{\pm 1\}^n$ define n_ξ to be 1 if ξ is a product of an even number of the e_i and -1 if it is a product of an odd number of the e_i. Clearly, we have $n_\xi = -n_{\xi g_\sigma}$ for all ξ and σ, so $\ker \partial$ is non-trivial and hence Y° is orientable. Since the number of connected components is 2^{n-k}, the kernel is isomorphic to $\mathbb{Z}^{2^{n-k}}$.

If there is no such basis of $\{\pm 1\}^n$, then there is some $g_\sigma \in \mathbb{O}$ which is a product of an even number of other elements in $\mathbb{O}$, for otherwise we can reduce $\mathbb{O}$ to a linearly independent set and then extend it to a basis of $\{\pm 1\}^n$. We get $g_\sigma = g_{\sigma_1} \cdots g_{\sigma_{2k}}$ and hence $1 = g_\sigma g_{\sigma_1} \cdots g_{\sigma_{2k}}$, so for every ξ we get

$$n_\xi \;=\; (-1)^{2k+1} n_{\xi g_\sigma g_{\sigma_1} \cdots g_{\sigma_{2k}}} \;=\; -n_\xi\,,$$

which implies that $n_\xi = 0$ and hence $\ker \partial = 0$ and so Y° is non-orientable. □

We restate the orientability criteria of Theorem 2.1 for toric varieties.

Corollary 2.2. *Let Y be a real toric variety defined by a fan Σ. Then Y° is orientable if and only if there exists a basis of $\{\pm 1\}^n$ such that $(-1)^v$ is a product of an odd number of basis vectors, for each primitive vector v lying on a ray of Σ.*

The condition of this corollary is easily checked.

Lemma 2.3. *Given $A \subset \{\pm 1\}^n$, the condition that there exists a basis of $\{\pm 1\}^n$ such that each vector in A is a product of an odd number of basis vectors, is equivalent to the condition that no product of an odd number of vectors in A is equal to 1 in $\{\pm 1\}^n$.*

Proof. If we had $v_1 \cdots v_{2k+1} = 1$, then $v_{2k+1} = v_1 \cdots v_{2k}$, and expressing each v_i as an odd product of the e_i's we get a contradiction. For the other implication, reduce A to a linearly independent set A' and then extend A' to a basis of $\{\pm 1\}^n$. If there were a vector in $A \setminus A'$ which is a product of an even number of vectors $v = v_1 \cdots v_{2k}$, we would have then had $v \cdot v_1 \cdots v_{2k} = 1$. □

We may check if the condition is satisfied by reducing A to a linearly independent set A' and checking if the vectors in $A \setminus A'$ are products of odd numbers of vectors in A'.

Remark 2.4. The analog of Corollary 2.2 for a spherical toric variety is simply that we replace the group $\{\pm 1\}^n$ by $\{\pm 1\}^{1+n}$ and the elements $(-1)^v$ for v a primitive generator of a ray of Σ by the element $(-1)^{(b \cdot F, b)}$ for b a primitive normal vector to a facet F of Δ. The number of connected components of Y^+_Δ is the order of the quotient group

$$\{\pm 1\}^{1+n} / \langle (-1)^{(b \cdot F, b)} \mid b \text{ is primitive normal vector to a facet } F \text{ of } \Delta \rangle\,.$$

3. Applications

We settle some questions of orientability left open in [8] and then explain the motivation for these results from the study of real solutions to systems of polynomials.

3.1. **Cross Polytopes.**

The cross polytope is the convex hull of the standard basis vectors $e_1, e_2, \dots, e_n$ and their negatives $-e_1, -e_2, \dots, -e_n$. Except when $n = 1$ the corresponding toric variety is singular. The rays of its normal fan are the vertices $(\pm 1, \pm 1, \dots, \pm 1)$ of the n-cube, and these all have the same image in $(\mathbb{Z}/2\mathbb{Z})^n$. Extending this common image to a basis of $(\mathbb{Z}/2\mathbb{Z})^n$ shows that the hypotheses of Theorem 2.1 hold. Thus the corresponding toric variety is orientble and its smooth points have 2^{n-1} connected components. Figure 1 displays the cross polytope when $n = 2$ and an embedding in $\mathbb{R}^3$ of the corresponding toric variety. This example was treated in detail in [9, § 7].

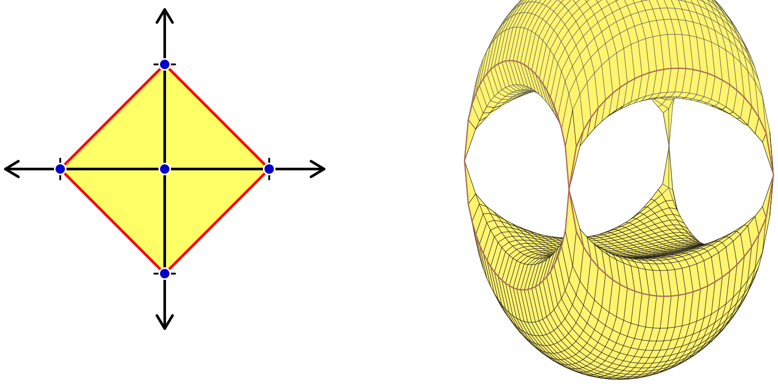

FIGURE 1. Two-dimensional cross polytope and double pillow.

3.2. **Order Polytopes.** The *order polytope $O(P)$* [10] of a finite poset P is

$$O(P) \ := \ \{y \in [0,1]^P \mid a \leq b \text{ in } P \ \Rightarrow \ y_a \leq y_b\}\,.$$

The integer points of $O(P)$ are its vertices and they correspond to the order ideals (upward-closed sets) of P.

Theorem 3.1. *The real toric variety $Y_{O(P)}$ is orientable if and only if all maximal chains of P have odd length.*

Proof. Lemma 4.9 of [8] (or rather its proof) implies that $Y_{O(P)}$ is orientable if all maximal chains of P have odd length. We establish the converse.

Facets of the order polytope have three types

$$\begin{array}{ll} y_a = 0 & \text{for } a \in P \text{ minimal,} \\ y_b = 1 & \text{for } b \in P \text{ maximal,} \\ y_b - y_a = 0 & \text{for } a \text{ covering } b \text{ in } P\,. \end{array}$$

If we replace each $=$ by $\geq$, these each give a valid inequality for $O(P)$, which we write in matrix form as $\mathcal{A}y \geq c$, for some vector c. In fact

$$O(P) \ := \ \{y \in | \ \mathcal{A}y \geq c\}\ . \tag{3.1}$$

By Corollary 2.2, $Y_{O(P)}$ is orientable if and only if there is a basis of the row space of $\mathcal{A}$, reduced modulo 2, such that each row is a sum of an odd number of basis vectors.

Fix a maximal chain $a_1 < \cdots < a_k$ in P. The corresponding facets of $O(P)$ are

$$y_{a_1} \ = \ 0\,, \quad y_{a_2} - y_{a_1} \ = \ 0\,, \quad \ldots, \quad y_{a_k} - y_{a_{k-1}} \ = \ 0\,, \quad y_{a_k} \ = \ 1\,,$$

and the corresponding rows of the matrix $\mathcal{A}$ (reduced modulo 2) are

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{bmatrix}$$

The sum of the rows that correspond to a fixed maximal chain of length k is equal to zero modulo 2, so we get a sum of $k+1$ rows equal to zero. If k is even, Lemma 2.3 implies that $Y_{O(P)}$ is non-orientable. $\square$

A poset P is *ranked modulo 2* if there is a function $P \to \{\pm 1\}$ such that adjacent elements take opposite signs. Equivalently, in each connected component, all maximal chains have the same parity, or every cycle in the Hasse graph of P is even. In [8], "ranked modulo 2" meant that all maximal chains in P have the same parity.

Theorem 3.2. *A real spherical toric variety $Y^+_{O(P)}$ is orientable if and only if P is ranked modulo* 2*.*

Lemma 4.9 of [8] shows that $Y^+_{O(P)}$ is orientable if all maximal chains in P have the same parity, both directions of the theorem are new.

Proof. The order polytope is defined by the facet inequalities (3.1). For a maximal chain $a_1 < \cdots < a_k$ in P, the corresponding rows of the augmented matrix $[\mathcal{A} : c]$ are

$$\left[\begin{array}{cccccccccc|c} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 & & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & & 1 \end{array}\right] .$$

Note that each row of $[\mathcal{A} : c]$ has the form $(b, b \cdot F)$ (modulo 2), where b is a primitive normal to a facet F of Δ. By Remark 2.4, $Y^+_{O(P)}$ is orientable if and only if there is a basis of the row space of $[\mathcal{A} : c]$, reduced modulo 2, such that each row of $[\mathcal{A} : c]$ is a sum of an odd number of basis vectors.

If P is not ranked modulo 2, then there a cycle in the Hasse graph of P with an odd number of edges. Summing up the vectors that correspond to these edges, we get an odd sum of vectors equal to zero, so $Y^+_{O(P)}$ is not orientable, by Lemma 2.3.

Suppose that P is ranked modulo 2. A linear relation modulo 2 among the rows of $[\mathcal{A} : c]$ is a sum of linear relations that come from adding up the rows corresponding to the edges in a cycle in P. Since each component of P is ranked modulo 2, all such cycles have an even number of edges, so the linear relations are of the form $v_1 + v_2 + \cdots + v_k = 0$ where k is even, which implies that $Y^+_{O(P)}$ is orientable, by Lemma 2.3. $\square$

3.2.1. *Real solutions to systems of equations.* In [8] we considered systems,

$$f_1(x_1, \dots, x_n) \;=\; f_2(x_1, \dots, x_n) \;=\; \cdots \;=\; f_n(x_1, \dots, x_n) \;=\; 0\,, \tag{3.2}$$

where each f_i is a real polynomial whose exponent vectors lie in $\Delta \cap \mathbb{Z}^n$, for a fixed lattice polytope Δ. The solutions to (3.2) may be expressed as a linear section $L \cap Y_\Delta$ of the projective toric variety Y_Δ corresponding to Δ. Here $L \subset \mathbb{RP}^\Delta$ is a linear subspace of codimension n. Projecting from a general codimension-one linear subspace E of L, we may realize the solutions to (3.2) as the fibers of a map

$$\pi_E \,:\, Y_\Delta \longrightarrow \mathbb{RP}^n\,,$$

to real projective space. If n is odd, then $\mathbb{RP}^n$ is orientable. If Y_Δ is also orientable, then fixing any orientation of $\mathbb{RP}^n$ and Y_Δ, the map π_E has a degree whose absolute value gives a lower bound on the cardinality of a fiber of π_E, in particular on the number of real solutions to the system (3.2).

More generally, we may lift this projection to the spherical toric varieties

$$\pi_E^+ \,:\, Y_\Delta^+ \longrightarrow \mathbb{S}^n\,.$$

If Y_Δ^+ is orientable, we may fix an orientation and the absolute value of the degree of π_E^+ is a lower bound on the number of solutions to the system (3.2).

In this paper, we characterized the orientability of Y_Δ and Y_Δ^+. To use this to obtain lower bounds on the number of real solutions to systems of polynomials, we need to develop better tools to compute the degrees of these maps π_E and π_E^+.

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