Generalized Witness Sets

Software and Applications in Numerical Algebraic Geometry
SIAM Meeting on Applied Algebraic Geometry, 4 August 2015

Frank Sottile
sottile@math.tamu.edu

Based on discussions with: D. Bates, J. Hauenstein, and A. Leykin.
Numerical Algebraic Geometry

The origins of numerical algebraic geometry were in numerical homotopy continuation, a method to find all isolated complex solutions to a system of $n$ equations in $n$ variables.

The subject rightly began around 2000, when Sommese, Verschelde, and Wampler developed the notion of a witness set.

A witness set is a data structure for representing and manipulating an algebraic variety using numerical algorithms on a computer.

Witness sets are the foundation for many geometrically appealing algorithms in numerical algebraic geometry.

It is appropriate to consider the analogy:

Witness set : Numerical Algebraic Geometry

\[ \iff \] Gröbner Basis : Symbolic Computation
Let $V \subset \mathbb{C}^n$ (or $\subset \mathbb{P}^n$) be a $k$-dimensional variety

A **witness set** for $V$ is a triple $(W, F, L)$ where

- $F = (f_1, \ldots, f_{n-k})$ are polynomials such that $V$ is a component of $F^{-1}(0)$
- $L = (\ell_1, \ldots, \ell_k)$ are general affine forms defining a general $(n-k)$-plane $L^{-1}(0)$
- $W = V \cap L^{-1}(0)$

By Bertini’s Theorem, $W$ is a collection of deg $V$ reduced points

Moving $L$ enables us to sample points of $V$, test for membership in $V$, and perform many other geometric constructions

May view a witness set as a concrete manifestation of André Weil’s notion of a general point
What is a Witness Set? II

$V \subset \mathbb{C}^n$ is a cycle whose class lies in the Chow group $A_k \mathbb{P}^n$ (Chow := algebraic cycles modulo rational equivalence, $\sim$)

$L$ is a general representative of the distinguished generator of $A_k \mathbb{P}^n$

$W = V \cap L \in A_0 V$ (localized intersection product) is a reduced zero-cycle on $V$ that witnesses the cap product $[V] \cap [L]$

By (Poincaré) duality, $W$ represents $V$. Specifically, in $A_k \mathbb{P}^n$,

$$[V] = W \cdot [L_k], \quad L_k \text{ a } k\text{-plane}$$

There are acceptable variations, using images of algebraic cycles in cohomology, or numerical equivalence,...

Frank Sottile, Texas A&M University
Suppose that \( X \) is a smooth variety of dimension \( n \) with finitely generated Chow groups satisfying Poincaré duality

Let \( \{ L_{i,k} \mid k = 0, \ldots, n, \ i = 1, \ldots, \beta_k \} \) be cycles such that \( \{[L_{i,k}] \mid i = 1, \ldots, \beta_k \} \) forms a basis for \( A_k X \)

We will also want that

- For every point \( x \) of \( X \) and \( i, k \), there is a cycle \( \Lambda \) rationally equivalent to \( L_{i,k} \) containing \( x \)
- For \( Y \subset X \) of codimension \( k \) and any \( i = 1, \ldots, \beta_k \), there is a cycle \( \Lambda \) rationally equivalent to \( L_{i,k} \) with \( Y \cap \Lambda \) is transverse

While apparently restrictive, projective spaces, Grassmannians, flag manifolds and products of these spaces all have these properties
Given such a variety $X$ and representatives $L_{i,k}$

Let $V$ be a subvariety of $X$ of dimension $n-k$.

A witness set for $V$ is a list of pairs

$$(W_1, \Lambda_1), \ldots, (W_{\beta_k}, \Lambda_{\beta_k})$$

where

- $\Lambda_i \sim L_{i,k}$ for $i = 1, \ldots, \beta_k$ with $\Lambda_i$ general
- $W_i = V \cap \Lambda_i$ is a transverse intersection (and is a set of reduced points)

(This may be modified to be more computational by including $n-k$ hypersurfaces (equations) $F$ whose intersection contains $V$ as a component, and also equations for the $\Lambda_i$)
Rational Equivalence. Suppose that $U \subset X \times \mathbb{C}$ is irreducible of dimension $k+1$ with $k$-dimensional fibers over $\mathbb{C}$ (the map $f$ to $\mathbb{C}$ is flat). Then $f^{-1}(0) \sim f^{-1}(1)$. These elementary rational equivalences generate $\sim$ on algebraic cycles.

Rational equivalence is just an algebraic homotopy.

Membership. Given $x \in X$ and a nonempty witness set $(W_i, \Lambda_i)$ for $V$. Let $\Lambda' \sim \Lambda_i$ contain $x$.

The chain of elementary rational equivalences gives a homotopy between $W_i = V \cap \Lambda_i$ and $W' := V \cap \Lambda'$.

Then $x \in V \iff x \in W'$.

Other algorithms also extend to this setting.
Examples

Grassmannians. The Grassmannian has distinguished Schubert varieties $X_\alpha F$ whose classes form a basis of its Chow ring, and satisfy duality

These cover the Grassmannian and satisfy a Bertini Theorem

Regeneration is also possible. The Picard group is $\mathbb{Z}$, so every hypersurface is a multiple of the Schubert divisor, $D$. The geometric Pieri rule (Schubert, 1884) gives an easy homotopy between

$$D \cap X_\alpha F \quad \text{and} \quad \sum_{\beta \prec \alpha} X_\beta F$$

Other varieties. These properties (except Pic, which is free abelian) hold for products of Grassmannians, including products of projective spaces: (See mss. of Hauenstein-Rodriguez on multiprojective varieties). Most are known to hold for flag manifolds.
Challenge: Implement and refine these ideas

Oeding: Does there exist a reasonable notion of an equivariant witness set?

References.