Bott-Samelson varieties and combinatorics

Laura Escobar

University of Illinois at Urbana-Champaign

Texas A&M

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Based on:
arXiv:1404.467
arXiv:1605.05613 (with O. Pechenik, B. Tenner, and A. Yong)
arXiv:1708.06663 (with B. Wyser and A. Yong)
Schur-Horn Theorem

\[ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n. \]

\( \mathcal{O}_\lambda := \{ \text{Hermitian matrices with eigenvalues } (\lambda_1, \ldots, \lambda_n) \}. \)

**Schur-Horn Theorem.** There is a matrix in \( \mathcal{O}_\lambda \) with diagonal entries \( (d_1, \ldots, d_n) \) if and only if \( (d_1, \ldots, d_n) \in \mathcal{P}_\lambda. \)

\( \mathcal{P}_\lambda := \text{conv}\{(\lambda_{w_1}, \ldots, \lambda_{w_n}) | w \text{ a permutation of } [n]\}. \)
Atiyah-Guillemin-Sternberg Convexity Theorem

Atiyah-Guillemin-Sternberg Convexity Theorem. Suppose that $M$ is a compact connected symplectic manifold with an action of a torus $T$ and moment map $\Phi : M \to \mathfrak{t}^*$ for this action. Then $\Phi(M) = \text{conv}\{\Phi(p) \mid p \text{ is a } T\text{-fixed point of } M\}$.

This polytope is called the moment polytope.
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**Schur-Horn Theorem.** There is a matrix in $O_\lambda$ with diagonal entries $(d_1, \ldots, d_n)$ if and only if $(d_1, \ldots, d_n) \in P_\lambda$.

$T = \{\text{diagonal matrices}\}$ acts on $O_\lambda$ by conjugation.

The $T$-fixed points are diagonal matrices.

The map $\Phi : O_\lambda \to \mathbb{R}^n$ defined by $\Phi(H) = (H_{11}, \ldots, H_{nn})$ is a moment map and $\Phi(O_\lambda) = P_\lambda$. 
Other moment polytopes

The Grassmannian $Gr(d, n)$ consists of the $d$-dimensional linear subspaces of $\mathbb{C}^n$.

The torus $T = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by component-wise multiplication.

$T$ acts on $Gr(d, n)$ by acting on the elements of a basis.

The moment map of $Gr(d, n)$ maps $V \in Gr(d, n)$ to the diagonal entries of the Hermitian matrix giving orthogonal projection onto $V$.

The image of the moment map for the Grassmannian is the hypersimplex.

**Theorem** (E.). The associahedron is the moment polytope of a submanifold of a product of Grassmannians.
Schubert varieties

The flag manifold is
Flag\(_n\) = \{(V_1, \ldots, V_n) : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\}
where each V\(_i\) is an i-dimensional vector subspace of \(\mathbb{C}^n\).

A Schubert variety consists of the (V\(_1\), \ldots, V\(_n\)) \(\in\) Flag\(_n\) satisfying some lower bounds on the dimension of V\(_i\) \(\cap\) C\(_j\) for all i, j.

Most Schubert varieties are not smooth.
Singularities of Schubert varieties

The **Kazhdan-Lusztig polynomial** $P_{v,w}$ measures how bad the singularity of the Schubert variety $X_w$ is at the point $e_v \in X_w$.

$X_w$ is smooth at $e_v$ if and only if $P_{v,w} = 1$.

The Kazhdan-Lusztig polynomials are the Poincaré polynomials for intersection homology of Schubert varieties.

All the coefficients are positive. There is no combinatorial proof of positivity.
Resolutions of singularities

If $Y$ is smooth then intersection homology is simply homology.

A **weak resolution** of $X$ is a polynomially defined surjective map $\pi : Y \to X$ such that $Y$ is smooth and $\pi$ is invertible at a dense open subset of $X$.

When $\pi$ is a small map, the intersection homology of $Y$ is isomorphic (as a group) to the intersection homology of $X$.

Resolutions of singularities provide tools to compute Kazhdan-Lusztig polynomials.

**Problem** (A. Zelevinsky ’83). Describe the Schubert varieties that admit small resolutions.
H.C. Hansen ’73 and M. Demazure ’74 independently presented the Bott-Samelson resolutions of Schubert varieties in the flag manifold.

A. Zelevinsky ’83 gave a resolution for Schubert varieties in the Grassmannian, presented as a configuration space of vector spaces prescribed by dimension and containment conditions.

P. Magyar ’98 gave a new description of the Bott-Samelson resolution in the same spirit.

In general, these resolutions are not small.
Preview of Main Results

**Theorem** (E., Pechenik, Tenner, Yong). The Bott-Samelson resolution of singularities of a Schubert variety consists of vector spaces arranged on a rhombic tiling.

Schubert varieties are B-orbit closures in the flag manifold. Replace B’s action with that of a symmetric group K to obtain symmetric orbit closures in the flag manifold.

**Theorem** (E.-Wyser-Yong). The Barbasch-Evens resolution of singularities of a symmetric orbit closure is a configuration space of vector spaces.

**Theorem** (E.-Wyser-Yong). The moment polytope of the Barbasch-Evens resolution is the convex hull of certain reflections of the moment polytope of the Bott-Samelson resolution.
Magyar’s construction of the Bott-Samelson manifold

\( I = (i_1, i_2, \ldots, i_k) \) where \( i_j \in [n] \).

The **Bott-Samelson manifold** is \( BS_I \subset \prod_{j=1}^{k} Gr(d_{i_j}, n) \).

\( BS_{(1,3,2,1)} = \{ (V_1, V_2, V_3, V_4) : \text{the following incidences hold} \} \subset Gr(1, 4) \times Gr(3, 4) \times Gr(2, 4) \times Gr(1, 4) \).
The Bott-Samelson map $\pi : BS_{(1,3,2,1)} \rightarrow \text{Flag}_4$ is

\[ \begin{array}{c}
\mathbb{C}^4 \\
\mathbb{C}^3 \quad V_2 \\
\mathbb{C}^2 \quad V_3 \\
\mathbb{C}^1 \quad V_1 \quad V_4 \\
0
\end{array} \rightarrow
\begin{array}{c}
\mathbb{C}^4 \\
V_2 \\
V_3 \\
V_1 \quad V_4 \\
0
\end{array} \]

The image of $\pi$ is a Schubert variety $X_w$.

If $\dim(BS_I) = \dim(X_w)$ then $\pi : BS_I \rightarrow X_w$ is a weak resolution of singularities.
Bott-Samelson resolutions and tilings

**Theorem** (E., Pechenik, Tenner, Yong). If \( \pi : BS_I \to X_w \) is a resolution of singularities, then \( BS_I \) consists of vector spaces arranged on a rhombic tiling of the Elnitsky polygon of a permutation \( w \).
Other resolutions

**Theorem** (E., Pechenik, Tenner, Yong). Given a zonotopal tiling $\zeta$ of the Elnitsky polygon of $w$, its corresponding generalized Bott-Samelson manifold $BS_\zeta$ together with the map $\pi_\zeta : BS_\zeta \to X_w$ is a resolution of singularities.
The action of $T$ on $Gr(d, n)$ induces an action of $T$ on $BS_I$.

**Proposition** (E., Pechenik, Tenner, Yong). The $T$-fixed points of $BS_I$ are in bijection with bipartitions of the rhombi.

$Gr(d, n)$ is a symplectic manifold and has moment map.

$BS_{(i_1, \ldots, i_k)}$ inherits a symplectic structure and moment map from $\prod_{j=1}^k Gr(d_{i_j}, n)$.

By the Atiyah-Guillemin-Sternberg convexity theorem $\Phi(BS_I)$ is the convex hull of the images of the $T$-fixed points.
Brick manifolds

Let $e_w$ be the only $T$-fixed general point of $X_w$.

The **brick manifold** $B_I$ is the fiber $\pi^{-1}(e_w)$ of the Bott-Samelson map $\pi : BS_I \to X_w$.
Let $e_w$ be the only $T$-fixed general point of $X_w$.

The **brick manifold** $B_I$ is the fiber $\pi^{-1}(e_w)$ of the Bott-Samelson map $\pi : BS_I \to X_w$.

\[
\begin{array}{c}
\mathbb{C}^3 \\
\langle e_1, e_2 \rangle \downarrow \\
\langle e_1 \rangle \downarrow \\
0
\end{array}
\quad
\begin{array}{c}
\langle e_3, e_2 \rangle \\
P_2 \\
\langle e_3 \rangle
\end{array}
\quad
\begin{array}{c}
\mathbb{C}^3 \\
\langle e_3, e_2 \rangle \\
\langle e_3 \rangle \\
0
\end{array}
\]

$B_{(1,2,1,2,1)} \ni C_3 \langle e_1 \rangle \langle e_2 \rangle \langle e_3 \rangle \rightarrow C_3 \langle e_3 \rangle = e_{321}$
The toric variety of the associahedron

\( B_I \) inherits a symplectic structure and moment map from \( BS_I \).

**Theorem** (E.). The moment polytope of the brick manifold is the brick polytope of V. Pilaud, F. Santos and C. Stump.

The associahedron is a polytope in which each vertex corresponds to a triangulation of a regular polygon.

Every associahedron of C. Hohlweg, C. Lange, and H. Thomas is the moment polytope of \( B_I \) for certain \( I \).

For these \( I \), \( \dim(B_I) = \dim(T) \). Therefore:

**Theorem** (E.). The toric variety of an associahedron of C. Hohlweg, C. Lange, and H. Thomas equals \( B_I \) for certain \( I \).
Loday associahedron

The brick manifold $B_{(1,2,3,1,2,3,1,2,1)}$ is the toric variety of Loday's 3D associahedron.
K-orbit closures

Let $\theta$ be an involution of $GL_n(\mathbb{C})$, and let $K$ be the subgroup of fixed points of the involution.

$K$ acts on $\text{Flag}_n$ with finitely many orbits.

Most $K$-orbit closures are not smooth.

For $K = GL_p \times GL_q$, where $p + q = n$, a $K$-orbit closure consists of $(V_1, \ldots, V_n) \in \text{Flag}_n$ satisfying some lower bounds on the dimensions of $V_i \cap \mathbb{C}^j$ and $V_i \cap (\mathbb{C}^j)^\perp$ for all $i, j$. 
Singularities of K-orbit closures

The **Kazhdan-Lusztig-Vogan polynomials** are a family of polynomials associated to a symmetric pair \((G, K)\).

They measure how bad the singularity of a K-orbit closure is at a point.

They are the Poincaré polynomials for intersection homology of K-orbit closures.

All the coefficients are positive. There is no combinatorial proof of positivity.

Resolutions of singularities provide tools to compute Kazhdan-Lusztig-Vogan polynomials.
D. Barbasch and S. Evens '94 presented resolutions for K-orbit closures analogous to the Bott-Samelson resolutions.

**Theorem** (E.-Wyser-Yong). A Barbasch-Evens variety for a symmetric pair \((G, K)\) is isomorphic as a K-variety to a configuration space of vector spaces prescribed by dimension and containment conditions.

We call this construction a **Barbasch-Evens-Magyar variety**.
Barbasch-Evens-Magyar varieties for $K = GL_p \times GL_q$

Fix $Y_0$ a closed $K$-orbit and $I = (i_1, \ldots, i_k)$ where $i_j \in [n]$.

The Barbasch-Evens-Magyar variety for $Y_0$ and $I$ is

$$BEM_{Y_0, I} \subset \prod_{j=1}^{n-1} Gr(j, n) \times \prod_{j=1}^{k} Gr(d_{i_j}, n).$$

For $Y_0 = \{(W_1, W_2, W_3) \in \text{Flag}_4 \mid W_2 = \mathbb{C}^2\}$ and $I = (2, 3)$,

$$BEM_{Y_0, (2,3)} =$$

$$\begin{array}{c}
\mathbb{C}^4 \\
\big\downarrow \\
W_3 \\
\big\downarrow \\
S_2 \\
\big\downarrow \\
P_1 \\
\big\downarrow \\
W_1 \\
\big\downarrow \\
0
\end{array}$$
Barbasch-Evens-Magyar varieties for $K = GL_p \times GL_q$

The map is $\pi : BEM_{Y_0,(2,3)} \to \text{Flag}_4$ is

\[
\begin{array}{c}
\mathbb{C}^4 \\
\downarrow \\
W_3 \downarrow S_2 \\
\downarrow \\
\mathbb{C}^2 \downarrow P_1 \\
\downarrow \\
W_1 \\
\downarrow \\
0
\end{array}
\quad \mapsto
\begin{array}{c}
\mathbb{C}^4 \\
\downarrow \\
S_2 \\
\downarrow \\
P_1 \\
\downarrow \\
W_1 \\
\downarrow \\
0
\end{array}
\]

The image of $\pi$ is a $K$-orbit closure $Y$.

If $w = s_{i_1} \cdots s_{i_k}$ is reduced, $\dim(Y) = \dim(BEM_{Y_0,l})$, and $Y$ is multiplicity-free then $\pi$ is a resolution of singularities.
Moment polytopes

\(BEM_{Y_0,I}\) inherits a symplectic structure and moment map from
\(\prod_{j=1}^{n-1} \text{Gr}(j, n) \times \prod_{j=1}^{k} \text{Gr}(d_{ij}, n)\).

**Theorem** (E.-Wyser-Yong). The moment polytope of \(BEM_{Y_0,I}\) is
the convex hull of certain \(S_n\)-reflections of the moment polytope of
\(BS_I\).

The moment polytope of \(BEM_{Y_0,(2,3)}\) is
the convex hull of four reflections in \(\mathbb{R}^3\)
of the moment polytope of the
Bott-Samelson variety \(BS_{(2,3)}\) (white).
Summary

The role of the permutahedron in the Schur-Horn theorem can be explained in terms of Hamiltonian symplectic manifolds and their moment maps.

Schubert varieties, and their analogues, the K-orbit closures are interesting singular varieties.

They have combinatorially described resolutions of singularities.

These descriptions allow one to deepen our understanding of the singular structure of Schubert varieties and K-orbit closures, and to study their moment polytopes.

Thank you!