(16) A great many of Euclid's propositions can be interpreted as constructions with straightedge and compass, although he never mentions those instruments explicitly.
(17) Euclid provided constructions for bisecting and trisecting any angle.
(18) Although π is a Greek letter, in Euclid's *Elements* it did not denote the number we understand it to denote today.

Exercises

In Exercises 1–4, you are asked to define some familiar geometric terms. The exercises provide a review of these terms as well as practice in formulating definitions with precision. In making a definition, you may use the five undefined geometric terms and all other geometric terms that have been defined in the text so far or in any preceding exercises.

Making a definition sometimes requires a bit of thought. For example, how would you define *perpendicularity* for two lines l and m? A first attempt might be to say that "l and m intersect and at their point of intersection these lines form right angles." It would be legitimate to use the terms "intersect" and "right angle" because they have been previously defined. But what is meant by the statement that *lines* form right angles? Surely, we can all draw a picture to show what we mean, but the problem is to express the idea verbally using only terms introduced previously. According to the definition on page 18, an angle is formed by two nonopposite *rays* emanating from the same vertex. We may therefore define l and m as *perpendicular* if they intersect at a point A and if there is a ray $\overrightarrow{AB}$ that is part of l and a ray $\overrightarrow{AC}$ that is part of m such that $\sphericalangle BAC$ is a right angle (Figure 1.17). We denote this by $l \perp m$.

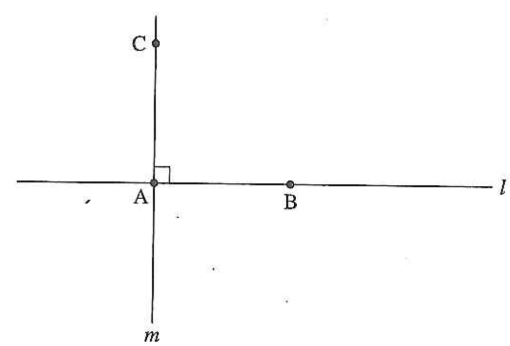

Figure 1.17 Perpendicular lines.

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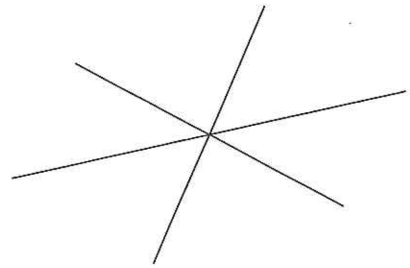


Figure 1.18 Concurrent lines.

1. Define the following terms:
   (a) *Midpoint* M of a segment AB.
   (b) *Perpendicular bisector* of a segment AB (you may use the term "midpoint" since you have just defined it).
   (c) Ray $\overrightarrow{BD}$ *bisects* angle $\sphericalangle$ABC (given that point D is between A and C).
   (d) Points A, B, and C are *collinear.*
   (e) Lines l, m, and n are *concurrent* (see Figure 1.18).
2. Define the following terms:
   (a) The *triangle* △ABC formed by three noncollinear points A, B, and C.
   (b) The *vertices, sides,* and *angles* of △ABC. (The "sides" are segments, not lines.)
   (c) The sides *opposite to* and *adjacent to* a given vertex A of △ABC.
   (d) *Medians* of a triangle (see Figure 1.19).
   (e) *Altitudes* of a triangle (see Figure 1.20).
   (f) *Isosceles* triangle, its *base,* and its *base angles.*
   (g) *Equilateral* triangle.
   (h) *Right* triangle.

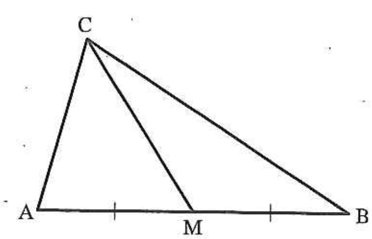


Figure 1.19 Median.

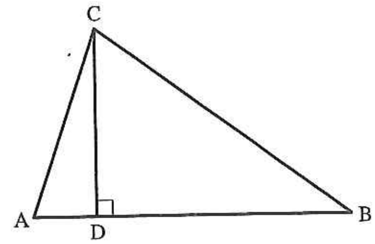


Figure 1.20 Altitude.

3. Given four points, A, B, C, and D, no three of which are collinear and such that any pair of the segments AB, BC, CD, and DA either have no point in common or have only an endpoint in common. We can then define the *quadrilateral* □ABCD to consist of the four segments mentioned, which are called its *sides*, the four points being called its *vertices* (see Figure 1.21). (Note that the order in which the letters are written is essential. For example, □ABCD may not denote a quadrilateral because, for example, AB might cross CD. If □ABCD did denote a quadrilateral, it would not denote the same one as □ACDB. Which permutations of the four letters A, B, C, and D do denote the same quadrilateral as □ABCD?) Using this definition, define the following notions:
   (a) The *angles* of □ABCD.
   (b) *Adjacent* sides of □ABCD.
   (c) *Opposite* sides of □ABCD.
   (d) The *diagonals* of □ABCD.
   (e) A *parallelogram*. (Use the word "parallel.")
4. Define *vertical angles* (Figure 1.22). How would you attempt to prove that vertical angles are congruent to each other? (Just sketch a plan for a proof—don't carry it out in detail.)
5. Use a common notion to prove the following result: If P and Q are any points on a circle with center O and radius OA, then OP ≅ OQ.

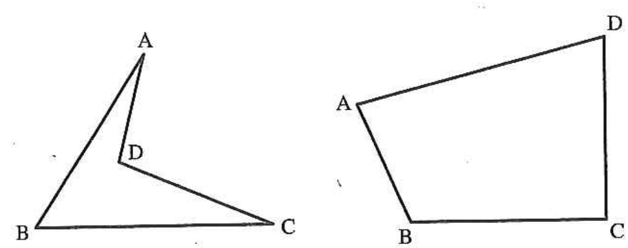
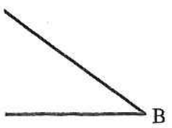


Figure 1.21 Quadrilaterals.



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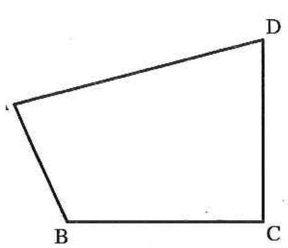

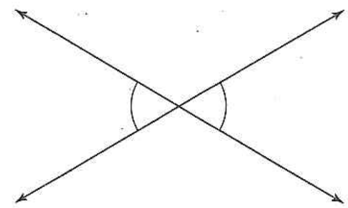

Figure 1.22 Vertical angles.

6. (a) Given two points A and B and a third point C between them. (Recall that "between" is an *undefined* term.) Can you think of any way to prove from the postulates that C lies on line $\overleftrightarrow{AB}$?
   (b) Assuming that you succeeded in proving C lies on $\overleftrightarrow{AB}$, can you prove from the definition of "ray" and the postulates that $\overrightarrow{AB} = \overrightarrow{AC}$?
7. If S and T are any sets, their *union* ($S \cup T$) and *intersection* ($S \cap T$) are defined as follows:
   (i) Something belongs to $S \cup T$ if and only if it belongs either to S or to T (or to both of them).
   (ii) Something belongs to $S \cap T$ if and only if it belongs both to S and to T.

   Given two points A and B, consider the two rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$. Draw diagrams to show that $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$ and $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$. What additional axioms about the undefined term "between" must we assume in order to be able to *prove* these equalities?
8. To further illustrate the need for careful definition, consider the following possible definitions of *rectangle*:
   (i) A quadrilateral with four right angles.
   (ii) A quadrilateral with all angles congruent to one another.
   (iii) A parallelogram with at least one right angle.

   In this book we will take (i) *as our definition.* Your experience with Euclidean geometry may lead you to believe that these three definitions are equivalent; sketch informally how you might prove that and notice carefully which theorems you are tacitly assuming. In hyperbolic geometry, these definitions give rise to three different sets of quadrilaterals (see Chapter 6).
9. Can you think of any way to prove from the postulates that for every line l
   (a) There exists a point lying on l?
   (b) There exists a point not lying on l?
10. Can you think of any way to prove from the postulates that the plane is nonempty, i.e., that points and lines exist? (Discuss with

your instructor what it means to say that mathematical objects, such as points and lines, "exist.")

11. Do you think that the Euclidean parallel postulate is "obvious"? Write a brief essay explaining your answer.
12. What is the flaw in the "proof" that all triangles are isosceles? (All the theorems from Euclidean geometry used in the argument are correct.)
13. If the number π is defined as the ratio of the circumference of any circle to its diameter, what theorem must first be proved to legitimize this definition? For example, if I "define" a new number φ to be the ratio of the area of any circle to its diameter, that would not be legitimate. Explain why not.
14. In this exercise, we will review several basic Euclidean constructions with a straightedge and compass. Such constructions fascinated mathematicians from ancient Greece until the nineteenth century, when all classical construction problems were finally solved.
    (a) Given a segment AB. Construct the perpendicular bisector of AB. (Hint: Make AB a diagonal of a rhombus, as in Figure 1.23.)
    (b) Given a line l and a point P lying on l. Construct the line through P perpendicular to l. (Hint: Make P the midpoint of a segment of l.)
    (c) Given a line l and a point P *not* lying on l. Construct the line through P perpendicular to l. (Hint: Construct isosceles triangle $\triangle$ABP with base AB on l and use (a).)
    (d) Given a line l and a point P not lying on l. Construct a line through P parallel to l. (Hint: Use (b) and (c).)
    (e) Construct the bisecting ray of an angle. (Hint: Use the Euclidean theorem that the perpendicular bisector of the base on an

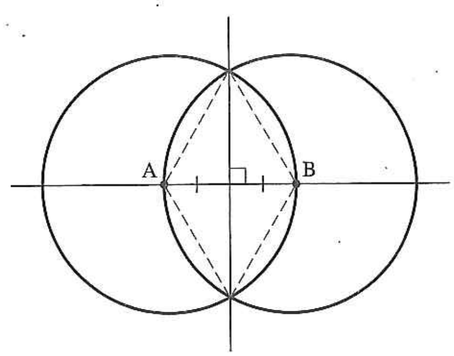

Figure 1.23

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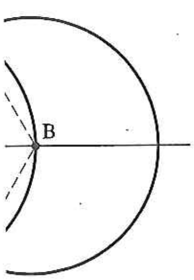

isosceles triangle is also the angle bisector of the angle opposite the base.)

(f) Given $\triangle ABC$ and segment $DE \cong AB$. Construct a point F on a given side of line $\overleftrightarrow{DE}$ such that $\triangle DEF \cong \triangle ABC$.

(g) Given angle $\sphericalangle ABC$ and ray $\overrightarrow{DE}$. Construct F on a given side of line $\overleftrightarrow{DE}$ such that $\sphericalangle ABC \cong \sphericalangle FDE$.

15. Euclid assumed the compass to be *collapsible.* That is, given two points P and Q, the compass can draw a circle with center P passing through Q (Postulate III); however, the spike cannot be moved to another center O to draw a circle of the same radius. Once the spike is moved, the compass collapses. Check through your constructions in Exercise 14 to see whether they are possible with a collapsible compass. (For purposes of this exercise, being "given" a line means being given two or more points on it.)

    (a) Given three points P, Q, and R. Construct with a straightedge and collapsible compass a rectangle $\square PQST$ with PQ as a side and such that $PT \cong PR$ (see Figure 1.24).

    (b) Given a segment PQ and a ray $\overrightarrow{AB}$. Construct the point C on $\overrightarrow{AB}$ such that $PQ \cong AC$. (Hint: Using part (a), construct rectangle $\square PAST$ with $PT \cong PQ$ and then draw the circle centered at A and passing through S.)

    Part (b) shows that you can transfer segments with a collapsible compass and a straightedge, so you can carry out all constructions as if your compass did not collapse.

16. The straightedge you used in the previous exercises was supposed to be *unruled* (if it did have marks on it, you weren't supposed to use them). Now, however, let us mark two points on the straightedge so as to mark off a certain distance d. Archimedes showed how we can then trisect an arbitrary angle.

    For any angle, draw a circle γ of radius d centered at the vertex O of the angle. This circle cuts the sides of the angle at points A and B. Place the marked straightedge so that one mark gives a

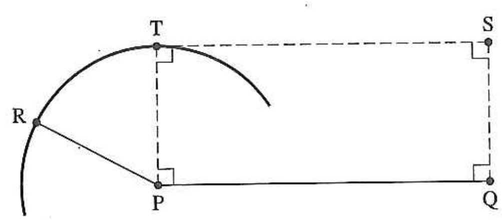

Figure 1.24