GROUP-THEORETICAL CLASSIFICATION OF POLYNOMIAL FUNCTIONS OF THE RIEMANN TENSOR

S. A. Fulling
Dept. of Mathematics, Texas A&M University, College Station, TX, 77843 USA

C. J. Cummins
Dept. of Math. and Statistics, Concordia University, Montreal, PQ, H3G 1M8 CANADA

R. C. King
Dept. of Mathematics, University of Southampton, Southampton SO9 5NH, U.K.

B. G. Wybourne
Dept. of Physics, University of Canterbury, Christchurch, NEW ZEALAND
and
Instytut Fizyki, Uniwersytet M. Kopernika, 87-100 Toruń, POLAND

1. Introduction. To motivate the kind of problem we are treating, consider the *heat kernel expansion* well known to researchers in general relativity, gauge theories, and differential geometry. Let $\Delta = g^{pq}(x)\nabla_p\nabla_q$ be the Laplace–Beltrami operator of a Riemannian manifold (with Riemann curvature tensor $R^a{}_{bcd}(x)$); let $K(t,x,y)$ be the integral kernel of the operator $e^{t\Delta}$ (which solves the initial value problem for $\partial\psi/\partial t = \Delta\psi$). Then the diagonal (coincidence) value of K has an asymptotic expansion at small t,

$$K(t,x,x) \sim (4\pi t)^{-d/2} \sum_{n=0}^{\infty} a_n(x) t^n$$

($d =$ dimension). The first two coefficients are well known:

$$a_1 = \frac{1}{6}\,R, \qquad a_2 = \frac{1}{30}\,\Delta R + \frac{1}{72}\,R^2 - \frac{1}{180}\,R^{pq}R_{pq} + \frac{1}{180}\,R^{pqrs}R_{pqrs}\,.$$

(Recall that the Ricci tensor is $R_{ab} \equiv R^p{}_{apb}$ and the curvature scalar is $R \equiv R^q{}_q$.) The third term is also known [e.g., 6]; it is a linear combination of 17 terms, of which R^3, $R^{pqr}{}_{;}R_{pr;q}$, $R^{pqrs}R_{pr;qs}$, and $\Delta^2 R$ are typical. It is clear that working to higher orders is giving rise to a combinatorial explosion.

Mathematicians and physicists have proposed a variety of algorithms for calculating a_n [e.g., 1, 2, 4, 6–7, 12, 13, 15, 17]. Advances in computer hardware and software are making high-order calculations increasingly practical. (*MathTensor*, the *Mathematica* tensor analysis program by Parker and Christensen [14], has been largely motivated by precisely this problem.) However, all methods eventually run into the same difficulty: combining a large number of similar terms into some comprehensible normal form. The symmetries of the Riemann tensor make this problem nontrivial. For example, $R^{pqrs}R_{prqs}$ is not linearly independent of $R^{pqrs}R_{pqrs}$, but this fact is not immediately obvious from consideration of the index symmetries of each factor separately. A different kind of example is provided by $R^p{}_q R^q{}_r R^r{}_s R^s{}_p$ — the trace of the fourth power of the Ricci tensor, regarded as a matrix. By a well known theorem of matrix theory, it is expressible as a polynomial in the lower-degree traces if $d < 4$.

Recognizing all such relationships, general and dimension-dependent, is a problem in group representation theory. The groups involved are S_n (the permutations of a tensor's indices), GL(d), and O(d). The methods required are known to physicists using group theory in atomic and nuclear physics [e.g., 18]. The lore is that associated with Young diagrams; indeed, the Young diagram representing the symmetries of R_{abcd} is the one with 4 blocks arranged in a square.

2. The basis problem for Riemann polynomials. Let us make the problem more precise with some formal definitions:

A *Riemann monomial* is an expression formed by tensor products and contractions from the Riemann tensor R and its covariant derivatives. A *Riemann polynomial* is a linear combination of these. (Actually, because of the rule relating commutation of covariant derivatives to R, we should work with *cosets* modulo terms of lower order and higher degree.)

Let $\mathcal{R}^r_{s,q}$ be the vector space of Riemann polynomials of *rank* r (number of free tensor indices), *degree* q (number of factors R), and *order* s (number of derivatives of $g =$ number of covariant derivatives plus twice q). Note that the heat kernel coefficient a_n belongs to $\bigoplus_{q=1}^n \mathcal{R}^0_{2n,q}$. We can further subdivide according to how the covariant derivatives are distributed among the factors; for example, $\mathcal{R}^0_{6,2} = \mathcal{R}^0_{\{2\,0\}} \oplus \mathcal{R}^0_{\{1\,1\}}$, where $R^{pqrs}R_{pr;qs}$ belongs to the first of these sets and $R^{pqr}{}_{;}{}^{s}R_{pr;q}$ to the second.

We can now state three increasingly ambitious versions of our problem: For $\mathcal{R}^r_{s,q}$,

(1) Find its dimension — the number of elements in a basis.

(2) Construct such a basis — list its elements. We want to choose the *best* basis — it should be "natural" or "simplest" or ....

(3) Provide a *normal form algorithm* — i.e., tell how to express an arbitrary element in terms of the basis.

In view of the nonuniqueness of the basis, one might add a fourth objective:

(4) Provide formulas or computer programs to convert from one basis to another.

3. Tools. The concepts employed include *irreducible representation*, *outer product*, *plethysm*, *branching rules*, *modification rules* [3, 8, 9–11, 16, 18]. (Since there is no space here for a course in group representation theory, we can only cite the jargon.) A major tool is the computer program `SCHUR` written by Wybourne and his students [19].

4. Results so far [5]**.** On objective (1): `SCHUR` easily provides us with the number of scalars through order 12. For example, in order 6 one gets the table

class	2	3	4	5	6	*total*
$\mathcal{R}^0_{6,1}$	1					1
$\mathcal{R}^0_{\{2\,0\}}$	1	2	1			4
$\mathcal{R}^0_{\{1\,1\}}$	1	2	1			4
$\mathcal{R}^0_{6,3}$	1	2	3	1	1	8
Total	4	6	5	1	1	**17**

where the column heading is the minimal dimension in which the object is independent of simpler ones. We find 92 scalars in order 8 (cf. [1]), 668 in order 10, and 6721 in order 12. (Since order is related to dimension in applications of a_n, these last are potentially relevant to Kaluza–Klein and string theories.)

On objective (2): We have lists of all the scalars through order 8 and all the higher rank tensors through order 6. For example, the table for $\mathcal{R}^5_{5,2}$ reads

tensor	*representation*	*dimension*
$R_{ab}R_{cd;e}$	$[5]+2[4\,1]+2[3\,2]+[3\,1^2]+[2^2 1]$	30
$R_{;a}R_{bcde}$	$[3\,2]+[2^2 1]$	10
$RR_{abcd;e}$	$[3\,2]$	5
$R^p{}_{a;b}R_{pcde}$	$[4\,1]+2[3\,2]+2[3\,1^2]+2[2^2 1]+[2\,1^3]$	40
$R_{ab;}{}^pR_{pcde}$	$[4\,1]+[3\,2]+[3\,1^2]+[2^2 1]$	20
$R^p{}_aR_{pbcd;e}$	$[4\,1]+2[3\,2]+[3\,1^2]+[2^2 1]$	25
$R^{pq}{}_{ab}R_{pqcd;e}$	$[4\,1]+[3\,2]+[3\,1^2]+[2^2 1]$	20
$R^p{}_a{}^q{}_bR_{pcqd;e}$	$[5]+[4\,1]+2[3\,2]+[3\,1^2]+[2^2 1]+[2\,1^3]$	30

The dimension stated is the number of independent index permutations, and the decomposition of the corresponding S_5 and $\mathrm{O}(d)$ representation into irreducibles is given.

Objectives (3) and (4) are implicit in the foregoing results, but not yet realized in practice. Their proper embodiment is in computer software, not a published document.

The methods shown here can be applied to problems involving other tensors in addition to R.

References

1. P. Amsterdamski, A. L. Berkin, and D. J. O'Connor, b_8 "Hamidew" coefficient for a scalar field, *Class. Quantum Grav.* **6**, 1981–1991 (1989).
2. I. G. Avramidi, A covariant technique for the calculation of the one-loop effective action, *Nucl. Phys. B* **355**, 712–754 (1991).
3. G. R. E. Black, R. C. King, and B. G Wybourne, Kronecker products for compact semisimple Lie groups, *J. Phys. A* **16**, 1555–1589 (1983).
4. B. S. DeWitt, *The Dynamical Theory of Groups and Fields*, Gordon and Breach, New York, 1965.
5. S. A. Fulling, C. J. Cummins, R. C. King, and B. G. Wybourne, Normal forms for tensor polynomials. I. The Riemann tensor, to appear.
6. P. B. Gilkey, The spectral geometry of a Riemannian manifold, *J. Diff. Geom.* **10**, 601–618 (1975).
7. P. B. Gilkey, Recursion relations and the asymptotic behavior of the eigenvalues of the Laplacian, *Compos. Math.* **38**, 201–240 (1979).
8. R. C. King, Branching rules for classical Lie groups using tensor and spinor methods, *J. Phys. A* **8**, 429–449 (1975).
9. D. E. Littlewood, Invariant theory, tensors and group characters, *Phil. Transac. Roy. Soc.* (London) *A* **239**, 305–365 (1944).
10. D. E. Littlewood, On invariant theory under restricted groups, *Phil. Transac. Roy. Soc.* (London) *A* **239**, 387–417 (1944).
11. D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, 2nd ed., Clarendon Press, Oxford, 1950.
12. S. Minakshisundaram and Å. Pleijel, Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds, *Canad. J. Math.* **1**, 242–256 (1949).
13. R. I. Nepomechie, Calculating heat kernels, *Phys. Rev. D* **31**, 3291–3292 (1985).
14. L. Parker and S. M. Christensen, *MathTensor: A System for Doing Tensor Analysis by Computer*, MathSolutions, Inc., Chapel Hill, N.C., 1991.
15. R. T. Seeley, Complex powers of an elliptic operator, *Singular Integrals (Proc. Sympos. Pure Math.* **10***)*, American Mathematical Society, Providence, R.I., 1967, pp. 288–307.
16. H. Weyl, *The Classical Groups*, Princeton University Press, Princeton, 1939.
17. H. Widom, A complete symbolic calculus for pseudodifferential operators, *Bull. Sci. Math.* **104**, 19–63 (1980).
18. B. G. Wybourne, *Symmetry Principles and Atomic Spectroscopy*, Wiley, New York, 1970.
19. [B. G. Wybourne and others], `SCHUR`*: An Interactive Programme for Calculating Properties of Lie Groups*, `SCHUR` Software Associates, Christchurch, N.Z., 1988.