Distributional Einstein equations for a Flat Wall

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Introduction

The first purpose of this talk is to review the cutoff calculations for flat walls and the distributional Einstein equations.

- Estrada et al., Vacuum Stress-Energy Density and Its Gravitational Implications ("Leipzig paper") J. Phys. A 41 (2008) 164055.
- Fulling et al., Energy Density and Pressure in Power-Wall Models ("Benasque paper") International Journal of Modern Physics: Conference Series.

The second purpose of the talk is to report our recent progress, starting with a visit in March to LSU to consult with Ricardo Estrada and Yunyun Yang.

Boundaries

For idealized boundary conditions, energy density has *nonintegrable* singularities near boundaries.

$$T_{00} \sim \frac{c_1}{s^4} + \frac{c_2}{s^3} + \cdots \qquad (s = \text{distance from boundary}). \tag{1}$$

Zeta-function regularization magically removes (most of) these infinities from the *total energy*. Ultraviolet-cutoff regularization requires them to be discarded ad hoc (with logarithmic ambiguity in cases where zeta has a pole).

This is all well and good for calculating forces between rigid bodies. But what about gravity? (Deutsch & Candelas 1979)

A Program

For realistic BC, these boundary effects are (large but) finite. They are part of the energy of the boundary material.

Working hypothesis: The stress tensor for idealized BC with the ultraviolet cutoff parameter *finite* is a reasonable ad hoc model for the true situation.

The theory will have a sensible renormalized limit when the cutoff is taken away. This requires making sense of the Einstein equation with a distributional source, "regularized" in the mathematical sense.

Gravitational effects in the lab are formally infinite but presumably actually tiny. Therefore, linearized Einstein equations should be OK. We take a flat background — but might need to add curvature due to mass of the boundary.

Scalar Field

$$S = \int_\Omega L \sqrt{g}\, d^{d+1}x, \tag{2}$$

$$L = \frac{1}{2}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R\phi^2], \qquad T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}\,. \tag{3}$$

$$\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \quad \text{with boundary conditions} \equiv -H\phi. \tag{4}$$

ξ labels different gravitational couplings. In the flat-space limit the field equation and (classical) total energy are independent of ξ, but the stress tensors are different.

Stress Tensor

$$T_{00}\left(\xi = \tfrac{1}{4}\right) = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - \phi \nabla^2 \phi\right], \tag{5}$$

$$T_{jj}\left(\xi = \tfrac{1}{4}\right) = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x_j}\right)^2 - \phi \frac{\partial^2 \phi}{\partial x_j^2}\right], \tag{6}$$

$$T_{\mu\nu}(\xi) = T_{\mu\nu}\left(\tfrac{1}{4}\right) + \Delta T_{\mu\nu}\,, \tag{7}$$

$$\Delta T_{00} = -2\left(\xi - \tfrac{1}{4}\right) \nabla \cdot (\phi \nabla \phi). \tag{8}$$

$$\Delta T_{jj} = -2\left(\xi - \tfrac{1}{4}\right)\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - \sum_{k \neq j}\left(\frac{\partial \phi}{\partial x_k}\right)^2 + \phi \frac{\partial^2 \phi}{\partial x_j^2}\right]. \tag{9}$$

We consider the scalar field, usually taking $\xi = \frac{1}{4}$.

$$\overline{T}(t, \mathbf{r}, \mathbf{r}') = -\sum_{n=1}^{\infty} \frac{1}{\omega_n} \phi_n(\mathbf{r}) \phi_n(\mathbf{r}')^* e^{-t\omega_n}. \tag{10}$$

$$\mathcal{E} = T_{00} = -\lim_{\cdots} \frac{1}{2} \frac{\partial^2 \overline{T}}{\partial t^2},$$

$$p_j = T_{jj} = \lim_{\cdots} \frac{1}{8} \left(\frac{\partial^2 \overline{T}}{\partial {x_j}^2} + \frac{\partial^2 \overline{T}}{\partial {x'_j}^2} - 2 \frac{\partial^2 \overline{T}}{\partial x \, \partial x'} \right).$$

Empty Space

$$\overline{T}_0(\mathbf{r},t;\mathbf{r}',t') = -\frac{1}{2\pi^2}\frac{1}{(t-t')^2+|\mathbf{r}-\mathbf{r}'|^2}\,.$$

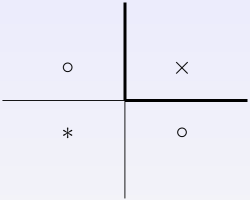
Note: t is a cutoff parameter, not the physical time (except under Wick rotation).

Local point-splitting in arbitrary direction u^μ (Christensen):

$$T_{\mu\nu} = \frac{1}{2\pi^2 t^4}\left(g_{\mu\nu} - 4\frac{u_\mu u_\nu}{u_\rho u^\rho}\right). \tag{11}$$

Thus $T^{\mu\nu}_{\text{ren}} = \Lambda g^{\mu\nu}$.

Two Walls Intersecting at a Right Angle



$\times$ = point under study

$\circ$ = reflection through a side

$*$ = reflection through a corner

$\bullet$ = periodic image (if any)

The multiple-reflection (image) method is *exact* in this case.

One Flat Hard Wall

Dirichlet wall at $z = 0$: $\quad (\mathbf{r}_\perp = (x, y))$

$$\overline{T}_{\text{ren}} = \frac{1}{2\pi^2} \frac{1}{t^2 + (\mathbf{r}_\perp - \mathbf{r'}_\perp)^2 + (z + z')^2}\,. \tag{12}$$

Set $\mathbf{r}_\perp = 0$ (and $t' = 0$).
Then t, $\mathbf{r}_\perp$, $z - z'$ are still available as cutoff parameters.

Recall $\mathcal{E} = -\frac{1}{2}\frac{\partial^2 \overline{T}}{\partial t^2}$, etc. Therefore ...

$$M \equiv t^2 + x^2 + y^2 + (z + z')^2.$$

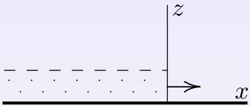
$$2\pi^2 \mathcal{E} = M^{-3}[-3t^2 + x^2 + y^2 + (z + z')^2],$$

$$2\pi^2 p_1 = M^{-3}[-t^2 + 3x^2 - y^2 - (z + z')^2],$$

$$p_2 \text{ similar;} \qquad p_3 = 0.$$

(Rigid displacement of the wall does not change the total energy. But there is a layer of energy against the wall.)

Insert a "Test" Wall



Imagine another planar boundary at $x = 0$; let's find pressure on it (from left side only). Volume of space occupied by boundary energy increases with x, so total energy does.

In accordance with the principle of energy balance (virtual work) one expects

$$F = \int_0^\infty T^{11}\,dz = -E = -\int_0^\infty T^{00}\,dz.$$

If *all cutoffs are removed*, $\mathcal{E} = \dfrac{1}{32\pi^2 z^4} = -p_1$, so energy balance is formally satisfied, but the integrals are divergent.

Point-Splitting

Ultraviolet Cutoff $(t \neq 0,\ \mathbf{r}_\perp = 0,\ z' = z)$

$$F = +\frac{1}{2}E \qquad (\text{not } (-1)E).$$

This E is negative and is the same one gets from expansion of

$$E = \frac{1}{2}\sum_n \omega_n e^{-t\omega_n}.$$

But we argue that this E is wrong and this F is (relatively) correct.

Point-splitting $\perp$ to movable wall $(x \neq 0$, others 0)

Then $(t, \mathcal{E})$ exchange places with $(x, -p_1)$.

$$F = +2E > 0.$$

(This time E is “right” and F is wrong.)

Point-splitting in neutral direction ($y \neq 0$, others 0)

$$F = -E, \qquad \text{as should happen!}$$

$$2\pi^2 \mathcal{E} = (y^2 + 4z^2)^{-2} > 0,$$

$$2\pi^2 p_1 = -(y^2 + 4z^2)^{-2}.$$

General ξ

The correction terms

- do not exhibit the paradox: $\Delta p = -\Delta \mathcal{E}$ always;
- integrate to 0 anyway.

Possible Responses to the Pressure Paradox

1. Divergent terms are so cutoff-dependent that they have no physical meaning whatsoever, and the only meaningful calculations are those in which these terms can be canceled out (e.g., forces between rigid bodies).

2. One must find a better model! (such as the soft wall)

3. Expressions with finite cutoff, such as $2\pi^2\mathcal{E} = (y^2 + 4z^2)^{-2}$ (where y is now a cutoff parameter, not a coordinate) can be regarded as ad hoc models of real materials, more physical and instructive than their limiting values, such as $\mathcal{E} = 1/32\pi^2 z^4$.

The paradox casts some doubt on the viability of this point of view. It now appears that physically plausible results can be obtained only by using different cutoffs for different parts of the stress tensor:

For the leading divergence (and higher-order divergences in the bulk that occur in curved space-time or external potentials) the preferred ansatz is "covariant point-splitting" based on the wave kernel, treating all directions in space-time equivalently, and removing the cutoff-dependent terms in such a way that the only ambiguity remaining can be regarded as a renormalization of the cosmological constant.

For the divergences at boundaries, it appears that the points must be separated parallel to the boundary, but in a direction orthogonal to the component of the stress tensor being calculated. Moreover, if the separation has a time component, a Wick rotation seems mandatory.

This situation cannot be regarded as a logically sound, long-term solution; its sole justification is that, unlike less contrived alternatives, it does not immediately produce results that are obviously wrong.

Isn't there an equal and opposite force from the other side of the wall?

In the flat case, maybe. Might not the BC be different on the other side?

In any case, this way out doesn't work for a spherical boundary, where the paradox was discovered (S.Fulling and M. Schaden). Ultraviolet cutoff gave $F = +\frac{1}{2}E$. Inside and outside energy layers have the same sign; total energy is proportional to surface area.

The wedge case will be discussed in the next talk. The cylindrical boundary case still needs to be investigated.

Linearized Einstein equation

In notation of Schutz's book, $-16\pi T_{\mu\nu} = \Box \bar{h}_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}(\mathrm{Tr}h)\eta_{\mu\nu} , \qquad h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} .$$

Assume the following three conditions:

- Linearized Einstein equation
- scalar field
- plane boundary (cutoff, approaching Dirichlet)

Assume static solution and let $\rho = T_{00}$, $h = h_{00}$.

$$-\nabla^2 h = 16\pi\rho = \pm\frac{8}{\pi}\,\frac{4x^2 - 3t^2}{(t^2 + 4x^2)^3}\,\theta(x).$$

Assume an infinite wall, so $\nabla^2 h = \frac{d^2 h}{dx^2}$.

Solution (homogeneous part ignored):

$$h(x) = \pm \frac{\theta(x)}{\pi} \left[\frac{4x}{t^3} \tan^{-1} \left(\frac{2x}{t} \right) - \frac{1}{t^2 + 4x^2} + \frac{1}{t^2} \right].$$

If we take the limit $t \downarrow 0$ in the equation, we get an ODE with a distribution as source. If we take the limit $t \downarrow 0$ in the solution, we get a singular distribution. Both limits involve somewhat arbitrary regularizations (Hadamard finite parts).

Moment expansion theorem

Theorem

Moment Expansion Theorem: *Let $f \in \mathcal{S}'(\mathbb{R})$ with support bounded on the left. Suppose*

$$f(x) = b_1 x^{\beta_1} + \cdots + b_n x^{\beta_n} + \mathcal{O}(x^{\beta}), \text{ as } x \to \infty, \tag{13}$$

where $\beta_1 > \beta_2 > \cdots > \beta_n > \beta$, and $-(k+1) > \beta > -(k+2)$. Then as $\lambda \to \infty$,

$$f(\lambda x) = \sum_{j=1}^{n} b_j g_j(\lambda x) + \sum_{j=0}^{k} (-1)^j \mu_j \frac{\delta^{(j)}(\lambda x)}{j!} + \mathcal{O}(\lambda^{\beta}) \tag{14}$$

in the space $\mathcal{S}'(\mathbb{R})$, where $g_j(x) = x^{\beta_j}\theta(x)$ if $\beta_j \neq -1, -2, -3, \ldots$ and $g_j(x) = \mathcal{P}f(x^{\beta_j}\theta(x))$ if $\beta_j = -1, -2, -3, \ldots$. Here the moments are

$$\mu_j(f) = \text{F.p.} \int_{-\infty}^{\infty} f(x) x^j \, dx. \tag{15}$$

This theorem describes in what technical conditions one can expand a test function ϕ in a Taylor series around $x = 0$ and then take the limit $\lambda \to \infty$ term by term ($\lambda = 1/t$ in our application).

Moment expansion theorem (intuitive summary): In certain distribution spaces, when a distribution $f(\lambda x)$ is applied to a test function ϕ, it is legitimate to expand ϕ in a Taylor series and take $\lambda \to \infty$ term by term, getting a series in $\delta^{(n)}(x)$.

Distributions and Hadamard finite part

References:

1. Ram P. Kanwal & Ricardo Estrada, *A Distributional Approach to Asymptotics: Theory and Applications*, 2002.
2. Estrada & Fulling, J. Phys. A **35** (2002) 3079.

A distribution f is a linear mapping from functions to numbers satisfying some technical conditions.
Ex. 1 : For a *function* f, $f[\phi] \equiv \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$.
Ex. 2 :$\delta[\phi] \equiv \phi(0)$. There is no function $\delta(x)$.
Precision requires a function space and a topology to define continuity of f. Calculus operations are defined by formal manipulations:

$$\begin{aligned} f'[\phi] &\equiv f[\phi'], \qquad (16) \\ \left(\int f(x,\alpha)\,d\alpha\right)[\phi] &= \int dx \int d\alpha\, f(x,\alpha)\phi(x) \equiv \int d\alpha \int dx f(x,\alpha)\phi(x) \end{aligned}$$

(applicable when $f = e^{ix\alpha}$, for example).

Hadamard finite part

What is $\int_{-\infty}^{\infty} \frac{1}{x^2}\phi(x)\,dx$?
Recall:

$$PV \int_{-\infty}^{\infty} \frac{1}{x}\phi(x)\,dx = \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} \frac{1}{x}\phi(x)\,dx + \int_{\epsilon}^{\infty} \frac{1}{x}\phi(x)\,dx\right] \tag{17}$$

and similarly for any *odd* power. (PV = "principal value".)

More generally, let $F(\epsilon) = \int_{|x|>\epsilon} f(x)\phi(x)\,dx = F_0(\epsilon) + F_1(\epsilon)$ where $F_0(\epsilon)$ is continuous at 0 and

$$F_1(\epsilon) = a_0 \ln \epsilon + \sum_{j=1}^{K} \frac{a_j}{\epsilon^j} \quad \text{for instance.} \tag{18}$$

Then you could define $f[\phi] \equiv F_0(0) \equiv F.p.f[\phi] \equiv \mathcal{P}\mathrm{f} f[\phi]$. (F.p. = "finite part" ; $\mathcal{P}\mathrm{f}$ = "pseudofunction".)

Example: Recall the Gelfand rule in dimensional regularization that

$$\int_0^\infty k^n \, dk = 0 \quad \forall n \in \mathbb{Z}.$$

Good news: F.p. $f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\, dx$ whenever the latter converges (i.e., whenever ϕ vanishes fast enough near 0). F.p. f is called a *regularization* of f.

Bad news: Scaling anomaly (for $H(x) =$ Heaviside step function):

$$\text{F.p.}\left(\frac{H(\lambda x)}{(\lambda x)^k}\right) = \frac{1}{\lambda^k}\,\text{F.p.}\left(\frac{H(x)}{x^k}\right) + \frac{\ln \lambda (-1)^{k-1} \delta^{(k-1)}(x)}{\lambda^k (k-1)!} \tag{19}$$

There is a similar problem for the derivative. (Every time k moves through an integer, we need to start discarding a new term.)

Hadamard $\mathcal{P}f$ formulas

$$\frac{d}{dx}[\theta(x)\ln x] = \mathcal{P}f\left(\frac{\theta(x)}{x}\right). \tag{20}$$

$$\frac{d}{dx}\mathcal{P}f\left(\frac{\theta(x)}{x^k}\right) = -k\mathcal{P}f\left(\frac{\theta(x)}{x^{k+1}}\right) + \frac{(-1)^k\delta^{(k)}(x)}{k!}\,. \tag{21}$$

$$\mathcal{P}f\left(\frac{\theta(\lambda x)}{(\lambda x)^{d+1}}\right) = \frac{1}{\lambda^{d+1}}\mathcal{P}f\left(\frac{\theta(x)}{x^{d+1}}\right) + \frac{(-1)^d\ln\lambda\delta^{(d)(x)}}{\lambda^{d+1}d!}. \tag{22}$$

$$\delta^{(j)}(\lambda x) = \lambda^{-(j+1)}\delta^{(j)}(x). \tag{23}$$

The first three Hadamard $\mathcal{P}f$ formulas are not scale-invariant; the ambiguous $\ln t$ terms result.

Back to the Einstein equation

By the moment expansion theorem, divergent leading powers can be discarded when using a singular function to define a distribution. More precisely, they are replaced by derivatives of δ with *arbitrary finite* coefficients.

Careful attention to the definitions shows that lim(soln) is indeed soln(lim) . In other words, our toy Einstein equation survives the renormalization process as a mathematically consistent differential equation.

$$\text{Equation :} \qquad -\nabla^2 h = 16\pi T_{00} = \frac{8}{\pi}\,\frac{4x^2 - 3t^2}{(t^2+4x^2)^3}\,\theta(x).$$

Solution:

$$h(x) = \frac{\theta(x)}{\pi}\left[\frac{4x}{t^3}\tan^{-1}\left(\frac{2x}{t}\right) - \frac{1}{t^2+4x^2} + \frac{1}{t^2}\right].$$

Limit equation: Let $\lambda = t^{-1}$.

$$-\frac{d^2 h_{00}(x)}{dx^2} = \frac{1}{2\pi}\ \text{F.p.}\left(\frac{\theta(x)}{x^4}\right) - 2\lambda^3\delta(x) + \frac{1}{\pi}\lambda^2\delta'(x) \tag{24}$$

$$+\frac{1}{8\pi}\delta'''(x) - \frac{1}{12\pi}\ln(2\lambda)\delta'''(x) + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{25}$$

It is indeed solved by the limit of the solution:

$$h_{00}(x) = 2\lambda^3\theta(x)x - \frac{1}{\pi}\lambda^2\theta(x) - \frac{1}{12\pi}\ \text{F.p.}\left(\frac{\theta(x)}{x^2}\right) \tag{26}$$

$$-\frac{1}{18\pi}\delta'(x) + \frac{1}{12\pi}\ln(2\lambda)\delta'(x). \tag{27}$$

The finite part is defined so that it integrates to 0; the operation is not scale-invariant, and that fact accounts for the apparent dimensional incoherence of the $\delta' \ln x_0$ term.

So far this is a review of the Leipzig calculations. Now we turn to the pressure and/or to neutral point-splitting.

Pressure

For the moment keep t as the cutoff parameter. The differential equation for the pressure component is

$$-\frac{d^2 h_{11}(z)}{dz^2} = 16\pi p_1 = \mp \frac{8}{\pi}\frac{1}{(t^2+4z^2)^2}\theta(z) \tag{28}$$

and the solution is given by

$$h_{11}(z) = \pm \frac{\theta(z)}{\pi}\left[\frac{2z}{t^3}\tan^{-1}\left(\frac{2z}{t}\right)\right]. \tag{29}$$

For the evaluation of the asymptotic behavior of (28), the relevant distribution for the moment expansion theorem is

$$f_1(z) = \frac{1}{(1+4z^2)^2}\theta(z) = \frac{1}{16}\frac{1}{z^4} + \mathcal{O}\left(\frac{1}{z^6}\right) \text{ as } z \to \infty. \tag{30}$$

Let's go back to the Benasque energy and pressure formulas:

$$M = t^2 + x^2 + y^2 + (z + z')^2.$$

$$2\pi^2 \mathcal{E} = M^{-3}[-3t^2 + x^2 + y^2 + (z + z')^2],$$

$$2\pi^2 p_1 = M^{-3}[-t^2 + 3x^2 - y^2 - (z + z')^2],$$

If t is the cutoff parameter, then

$$2\pi^2 p_1 = -M^{-3}[t^2 + 4z^2].$$

Now we take the distributional limit as $t \to \infty$.

The moment expansion theorem states, up to the relevant order, that the asymptotic expansion of $f_1(z)$ is

$$\begin{aligned} f_1(\lambda z) &\sim \tfrac{1}{16}\mathcal{P}f\left(\tfrac{\theta(\lambda z)}{(\lambda z)^4}\right) + \textstyle\sum_{j=0}^{3}(-1)^j \mu_j(f_1)\tfrac{\delta^{(j)}(\lambda z)}{j!} + \mathcal{O}\left(\tfrac{1}{\lambda^5}\right) \\ &= \tfrac{1}{16}\left\{\tfrac{1}{\lambda^4}\mathcal{P}f\left(\tfrac{\theta(z)}{z^4}\right) - \tfrac{\ln\lambda}{3!\lambda^4}\delta'''(z)\right\} \\ &+ \textstyle\sum_{j=0}^{3}(-1)^j\mu_j(f_1)\tfrac{\delta^{(j)}(z)}{j!\lambda^{j+1}} + \mathcal{O}\left(\tfrac{1}{\lambda^5}\right), \end{aligned} \tag{31}$$

and the moments $\mu_j(f_1)$ of the function f_1 are

$$\mu_0(f_1) = \int_0^\infty \frac{1}{(1+4z^2)^2}\,dz = \frac{\pi}{8}, \tag{32}$$

$$\mu_1(f_1) = \int_0^\infty \frac{1}{(1+4z^2)^2}\cdot z\,dz = \frac{1}{8}, \tag{33}$$

$$\mu_2(f_1) = \int_0^\infty \frac{1}{(1+4z^2)^2}\cdot z^2\,dz = \frac{\pi}{32}, \tag{34}$$

$$\mu_3(f_1) = \text{ F.p.}\int_0^\infty \frac{1}{(1+4z^2)^2}\cdot z^3\,dz = -\frac{1}{32} + \frac{1}{16}\ln 2. \tag{35}$$

The distributional limit of the differential equation (28) is

$$-\frac{d^2h_{11}(z)}{dz^2} = \mp\tfrac{1}{2\pi}\mathcal{P}f\left(\tfrac{\theta(z)}{z^4}\right) \mp \lambda^3\delta(z) \pm \tfrac{1}{\pi}\lambda^2\delta'(z) \pm \tfrac{1}{8}\lambda\delta''(z)$$
$$\mp\tfrac{1}{24\pi}\delta'''(z) \pm \tfrac{1}{12\pi}\ln(2\lambda)\delta'''(z) + \mathcal{O}\left(\tfrac{1}{\lambda}\right). \qquad (36)$$

The relevant functions for the analysis of $h(z)$ are

$$f_2(z) = z\tan^{-1}(2z)\theta(x) = \frac{\pi}{2}z - \frac{1}{2} + \frac{1}{24}\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^4}\right) \text{ as } z \to \infty. \qquad (37)$$

The moment expansion theorem says that

$$f_2(\lambda z) \sim \frac{\pi}{2}\theta(\lambda z)(\lambda z) - \frac{1}{2}\theta(\lambda z) + \frac{1}{24}\mathcal{P}f\left(\frac{\theta(\lambda z)}{(\lambda z)^2}\right) \qquad (38)$$
$$+ \sum_{j=0}^{1}(-1)^j\mu_j(f_2)\frac{\delta^{(j)}(\lambda z)}{j!} + \mathcal{O}\left(\frac{1}{\lambda}\right)$$
$$= \frac{\pi}{2}\lambda\theta(z)z - \frac{1}{2}\theta(z) + \frac{1}{24\lambda^2}\mathcal{P}f\left(\frac{\theta(z)}{z^2}\right) - \frac{1}{24}\frac{1}{\lambda^2}\ln\lambda\delta'(z)$$
$$+ \sum_{j=0}^{1}(-1)^j\mu_j(f_2)\frac{\delta^{(j)}(z)}{j!\lambda^{j+1}} + \mathcal{O}\left(\frac{1}{\lambda}\right).$$

Finally, the relevant moment expansion coefficients this time are

$$\mu_1(f_2) = \int_0^\infty z \tan^{-1}(2z)\, dz = \frac{\pi}{16}, \tag{39}$$

$$\mu_2(f_2) = \int_0^\infty z^2 \tan^{-1}(2z)\, dz = \frac{1}{72} + \frac{1}{24}\ln 2. \tag{40}$$

By forming the correct linear combination of these terms, we have

$$\begin{aligned} h_{11}(z) &= \pm\lambda^3\theta(z)z \mp \frac{1}{\pi}\lambda^2\theta(z) \pm \frac{1}{12\pi}\mathcal{P}f\left(\frac{\theta(z)}{z^2}\right) \mp \frac{1}{8}\lambda\delta(z) \\ &\mp \frac{2}{\pi}\left[\frac{1}{72} + \frac{1}{24}\ln 2\right]\delta'(z) \mp \frac{1}{12\pi}\ln\lambda\delta'(z) \\ &= \pm\lambda^3\theta(z)z \mp \frac{1}{\pi}\lambda^2\theta(z) \pm \frac{1}{12\pi}\mathcal{P}f\left(\frac{\theta(z)}{z^2}\right) \mp \frac{1}{8}\lambda\delta(z) \pm \frac{1}{36\pi}\delta'(z) \\ &\mp \frac{1}{12\pi}\ln(2\lambda)\delta'(z). \end{aligned} \tag{41}$$

By taking the second derivative of (41), according to the Hadamard $\mathcal{P}f$ formulas, we find that the equation (36) is satisfied.

Neutral point splitting

Now consider using y as the cutoff parameter. It is "neutral" with respect to both time (energy) and x (pressure p_1).

$$2\pi^2\mathcal{E} = M^{-3}[-3t^2 + x^2 + y^2 + (z+z')^2],$$

$$2\pi^2 p_1 = M^{-3}[-t^2 + 3x^2 - y^2 - (z+z')^2]$$

Note:

1. The p_1 calculation is identical to what we just did, with y in the role of t.
2. $\mathcal{E}$ in this case is just the negative of p_1.

So we do not need to do any more calculating, and we have 3 new, agreeing formulas that outvote the Leipzig formula for $\mathcal{E}$.

Comparison of old and new energy formulas

$$h_{00}^{\text{old}}(x) = \pm 2\lambda^3\theta(x)x \mp \frac{1}{\pi}\lambda^2\theta(x) \mp \frac{1}{12\pi}\mathcal{P}f\left(\frac{\theta(x)}{x^2}\right) \mp \frac{1}{18\pi}\delta'(x) \pm \frac{1}{12\pi}\ln(2\lambda)\delta'(x) \tag{42}$$

$$h_{00}^{\text{new}}(z) = \mp\lambda^3\theta(z)z \pm \frac{1}{\pi}\lambda^2\theta(z) \mp \frac{1}{12\pi}\mathcal{P}f\left(\frac{\theta(z)}{z^2}\right) \pm \frac{1}{8}\lambda\delta(z) \mp \frac{1}{36\pi}\delta'(z) \pm \frac{1}{12\pi}\ln(2\lambda)\delta'(z). \tag{43}$$

Here h_{00}^{new} is the negative of the pressure expression we obtained. ("old" $= t$ cutoff, "new" $= y$ cutoff.) For the ratios of the coefficients we find:

Table: Ratio $h_{00}^{\text{old}} / h_{00}^{\text{new}}$

Term	Ratio of the coefficients
$\lambda^3 x\theta(x)$	-2
$\lambda^2\theta(x)$	-1
$\lambda\delta(x)$	0
$\ln(2\lambda)\delta'(x)$	1
$\delta'(x)$	2
$\mathcal{P}f\left(\frac{\theta(x)}{x^2}\right)$	1

The first two terms could be absorbed into the homogeneous solution.

Philosophies

Two philosophies

1. The cutoff is a mathematical means to an end, which is a limiting theory with the cutoff removed. At the intermediate stage a violation of energy-pressure balance may be tolerated, so long as the final theory is physically acceptable. Noncovariant regularization forces noncovariant counterterms, which appear in the final equations with coefficient 0 (i.e., not at all).

2. The cutoff theory should be a physically plausible model of a real boundary. Energy-pressure balance must be preserved. The distributional limit is an approximation to the cutoff theory, not vice versa.

Open to discussion

- Are both philosophies physically tenable?
- What is the physical significance of the delta terms?
- Can they be taken seriously (in the neutral case) with λ finite?
- Should we be worried that the coefficient ratio is not unity for one term that's independent of λ?