Adventures Near a Sphere

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Goals:

- Study vacuum stress near a curved surface by classical-path analysis.
- Resolve discrepancy between energy and pressure calculations by partial-wave analysis (for scalar field, with exponential ultraviolet cutoff).

All constants and signs are preliminary and subject to change. (So are many conclusions.)

"The whole semiclassical approximation is only of pedagogical value in the case of a sphere, since the exact spectral representation of the Green function of a massless particle excluded from a spherical region is known."

Schaden & Spruch, Ann. Phys. **313** ('04) 37

The paradox

Divergent surface energy $E \propto (4\pi a^2)t^{-3}$ (same inside and out).

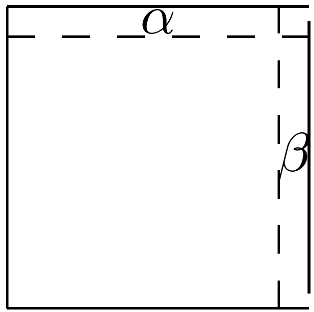
Generalized force $F = -\dfrac{\partial E}{\partial a} \propto -2a(4\pi)t^{-3}$.

Radial pressure $p_r = \dfrac{F}{4\pi a^2} \propto -\dfrac{2}{a}\, t^{-3}$.

The constant is (probably) $\dfrac{1}{8\pi} \times 2$, certainly not 0.

But

- Milton [PRD **68** ('03) 065020] following Bender and Milton [PRD **50** ('94) 6547] finds *no* divergent pressure. No obvious a^{-1} term even at mode sum level!
- Fulling et al. [0806.2468] in rectangle find no divergent pressure with divergent energy next to a side; divergence goes with energy near perpendicular side! In sphere, there is no perpendicular side.



Integral kernels in billiards

Heat: $K(t,\mathbf{r},\mathbf{r}') = \langle \mathbf{r}|e^{-tH}|\mathbf{r}'\rangle.$

$$K_0 = (4\pi t)^{-3/2} e^{-|\mathbf{r}-\mathbf{r}'|^2/4t}.$$

Quantum: $U = \langle e^{-itH}\rangle. \quad U_0 = (4\pi it)^{-3/2} e^{-|\mathbf{r}-\mathbf{r}'|^2/4t}.$

Cylinder: $\overline{T} = \left\langle \frac{e^{-t\sqrt{H}}}{-\sqrt{H}} \right\rangle. \quad \overline{T}_0 = -\frac{1}{2\pi^2}\frac{1}{t^2 + |\mathbf{r}-\mathbf{r}'|^2}.$

Resolvent: $G(k,\mathbf{r},\mathbf{r}') = \langle \mathbf{r}|(H-k^2)^{-1}|\mathbf{r}'\rangle.$

$$G_0 = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

(also wave kernels and zeta function)

Eigenvalue density $\sigma(k)\,dk = \dfrac{1}{\pi}\operatorname{Im} G(\mathbf{r},\mathbf{r}',k)\,d(k^2)$.

Energy density $\rho = -\dfrac{1}{2}\dfrac{\partial^2}{\partial t^2}\overline{T} \qquad (\text{if } \xi = \frac{1}{4})$.

Laplace transform in k^2: $\sigma \to K, \quad K \to G$.

Laplace transform in k: $\sigma \to \overline{T}$.

$$\int_0^\infty t^{-1/2} e^{-\tau^2/4t} K(t)\,dt = -\sqrt{\pi}\,\overline{T}(\tau)$$

because $\overline{T}$ is G (with $k=0$) in one higher dimension, or

$$\int_0^\infty t^{-1/2} e^{-\tau^2/4t} e^{-k^2 t}\,dt = \frac{\sqrt{\pi}}{k}\,e^{-\tau k}.$$

Synge–DeWitt formalism: [Christensen, PRD **14** ('76) 2490; Molzahn et al., *Ann. Phys.* **204** ('90) 64]

$l(\mathbf{r}, \mathbf{r}', y) \equiv$ distance from $\mathbf{r}'$ to $\mathbf{r}$ along (say) a straight path with specular reflections.

$\sigma(\mathbf{r}, \mathbf{r}') \equiv \frac{1}{2} l(\mathbf{r}, \mathbf{r}')^2$. Then $(\nabla \equiv \nabla_{\mathbf{r}})$

(1) $\nabla\sigma = l\nabla l = l\hat{n}$, $\hat{n} \equiv$ unit vector at $\mathbf{r}$ in the direction of the path;

(2) $(\nabla\sigma)^2 = l^2 = 2\sigma$;

(3) $\nabla^2\sigma = 1 + l\nabla \cdot \hat{n} = d + O(l^2)$ ($d = 3$ today);

(4) For the *direct* path,

$$\sigma = \tfrac{1}{2}|\mathbf{r} - \mathbf{r}'|^2, \quad \nabla\sigma = \mathbf{r} - \mathbf{r}', \quad \nabla^2\sigma = d.$$

Optical approximation: We attempt to construct (approximately) the kernel for the billiard problem in the form of a sum over specularly reflecting paths of terms

$$G_j = (-1)^j D_j(\mathbf{r}, \mathbf{r}') F(\sigma(\mathbf{r}, \mathbf{r}')),$$

where $(-1)^j$ means the parity of the number reflections (so that the Dirichlet condition is satisfied by the sum), and D does *not* depend on the parameter t or k (though F does).

Claim: D **comes out the same for all the kernels.** (for billiards!)

Proof and construction

- Plug ansatz into PDE.
- Group terms by order of singularity (for $\overline{T}$, powers of $(t^2 + 2\sigma)^{-1}$).
- Leading term vanishes if $F(\frac{1}{2}|\mathbf{r} - \mathbf{r}'|) = G_0(\cdot, \mathbf{r}, \mathbf{r}')$. ($F$ depends on t or k.)
- Next term vanishes if $D \equiv |\det M|^{1/2}$,

$$M_{jk} \equiv \frac{\partial^2 \sigma}{\partial r_j \, r'_k} \, .$$

$\det M = l^{d-1}\Delta, \Delta(\mathbf{r}, \mathbf{r}') \equiv$ enlargement factor $\frac{d(\text{angle})}{d(\text{area})}$ [Scardicchio & Jaffe, NPB **704** ('05) 552].

Energy and pressure in spherical symmetry

$$\rho = -\langle T^0_0 \rangle = -\frac{1}{2}\frac{\partial^2 \overline{T}}{\partial t^2} + \beta\left[\frac{\partial^2 \overline{T}}{\partial r\,\partial r'} + \frac{\partial^2 \overline{T}}{\partial r^2} + \frac{2}{r}\frac{\partial \overline{T}}{\partial r}\right].$$

$$p_r = \langle T^r_r \rangle = -\frac{1}{4}\left[\frac{\partial^2 \overline{T}}{\partial r\,\partial r'} - \frac{\partial^2 \overline{T}}{\partial r^2}\right] - \frac{2\beta}{r}\frac{\partial \overline{T}}{\partial r}.$$

$$p_\perp = \langle T^\theta_\theta \rangle = \langle T^\varphi_\varphi \rangle = \qquad [\beta \equiv \xi - \tfrac{1}{4}]$$

$$-\frac{1}{2r^2}\frac{\partial^2 \overline{T}}{\partial\theta\,\partial\theta'} + \frac{1}{4r}\frac{\partial \overline{T}}{\partial r} - \beta\left[\frac{\partial^2 \overline{T}}{\partial r\,\partial r'} + \frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r}\right].$$

[Cf. Schwartz-Perlov & Olum, PRD **72** ('05) 065013; Cavero-Peláez et al., PRD **73** ('06) 085004.]

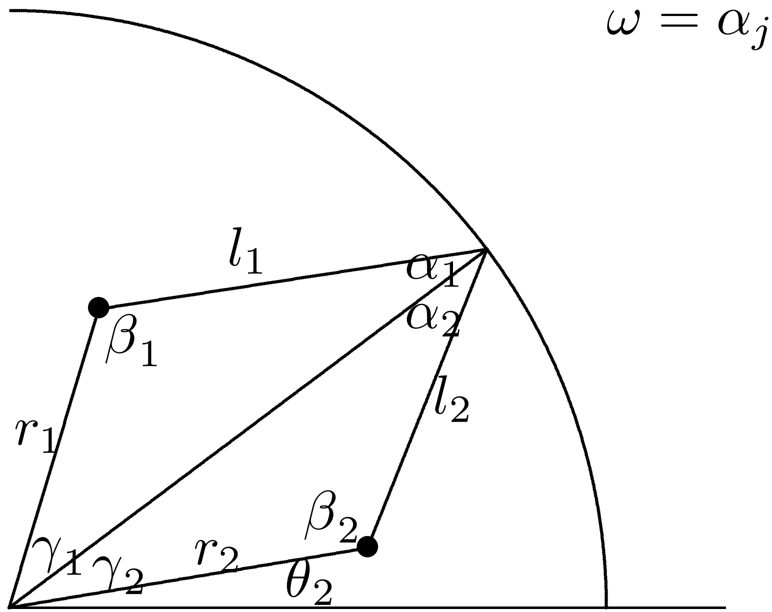
The geometrical quantities

Sometimes write r_1 and r_2 instead of r and r', etc. Ignore factor $-1/2\pi^2$. $\overline{T}$ will mean the single-reflection term to be subtracted from $\overline{T}_0$ in Dirichlet case. From S&J ($\pm \equiv$ (out / in), $\omega \equiv$ reflection angle)

$$\Delta = \left(\frac{l \pm 2l_1 l_2}{a\cos\omega}\right)^{-1} \left(\frac{l \pm 2l_1 l_2 \cos\omega}{a}\right)^{-1} .$$

$l = l_1 + l_2$. When $\mathbf{r}' = \mathbf{r}$,

$$l = 2|r-a| = \pm 2(r-a), \qquad \Delta = l^{-2}\left(1 \pm \frac{l}{2a}\right)^{-2} .$$


$\omega = \alpha_j$
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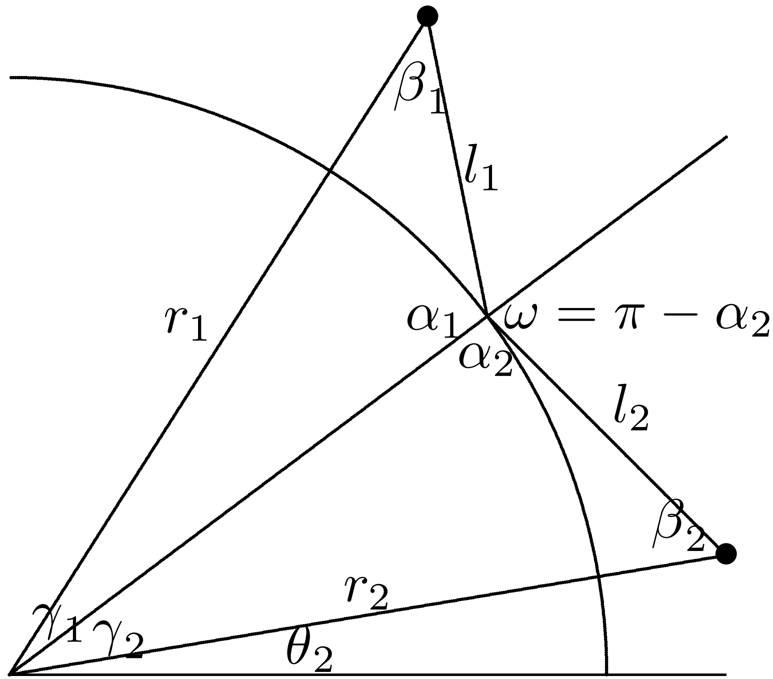
β_1
l_1
r_1
α_1
$\omega = \pi - \alpha_2$
α_2
l_2
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r_2
γ_1
γ_2
θ_2

Let $\Delta\theta \equiv \theta_1 - \theta_2 > 0$, $u \equiv \sin\omega$. Solve the triangles:

Interior:

$$l_j = a\sqrt{1-u^2} - \sqrt{r_j^2 - a^2u^2},$$

$$\Delta\theta = -2\sin^{-1} u + \sin^{-1}\left(\frac{au}{r_1}\right) + \sin^{-1}\left(\frac{au}{r_2}\right).$$

Exterior:

$$l_j = -a\sqrt{1-u^2} + \sqrt{r_j^2 - a^2u^2},$$

$$\Delta\theta = 2\sin^{-1} u - \sin^{-1}\left(\frac{au}{r_1}\right) - \sin^{-1}\left(\frac{au}{r_2}\right).$$

Proceed by implicit differentiation. (Signs checked for interior case only.)

$$B \equiv \frac{\partial \Delta\theta}{\partial u} = -\frac{2}{\sqrt{1-u^2}} + \frac{a}{\sqrt{r_1^2 - a^2u^2}} + \frac{a}{\sqrt{r_2^2 - a^2u^2}} .$$

$$\frac{\partial u}{\partial \theta_1} = -\frac{\partial u}{\partial \theta_2} = \frac{1}{B}, \qquad \frac{\partial u}{\partial r_j} = \frac{au}{B r_j \sqrt{r_j^2 - a^2u^2}} .$$

Mathematica from then on!

Derivatives of σ*:* If $u = 0$ (path perpendicular to sphere),

$$\frac{\partial\sigma}{\partial\theta_j} = 0, \qquad \frac{\partial\sigma}{\partial r_j} = r_1 + r_2 - 2a = -l,$$

$$\frac{\partial^2\sigma}{\partial r_1\,\partial r_2} = 1 = \frac{\partial^2\sigma}{\partial r_j^2}\,, \qquad \frac{\partial^2\sigma}{\partial r_1\,\partial\theta_2} = 0, \quad \text{etc.},$$

$$\frac{\partial^2\sigma}{\partial\theta_1\,\partial\theta_2} = -\,\frac{\partial^2\sigma}{\partial\theta_j^2} = r_1 r_2 a\,\frac{r_1 + r_2 - 2a}{a(r_1 + r_2) - 2r_1 r_2}\,.$$

When $r = r'$, $$\frac{\partial^2\sigma}{\partial\theta_1\,\partial\theta_2} = -\,\frac{r_1 r_2 a}{r} = -ra.$$

Derivatives of D: If $u = 0$,

$$D = \frac{2a^2 - ar_1 - ar_2}{ar_1 + ar_2 - 2r_1 r_2} \,.$$

("$\longrightarrow$" indicates setting both $r_j = r$)

$$\frac{\partial D}{\partial r_1} = -\,\frac{2a(a - r_2)^2}{(-2r_1 r_2 + a(r_1 + r_2))^2} \longrightarrow -\,\frac{a}{2r^2} \,,$$

$$\frac{\partial^2 D}{\partial r_1^2} = \frac{4a(a - 2r_2)(a - r_2)^2}{(-2r_1 r_2 + a(r_1 + r_2))^3} \longrightarrow \frac{a(a - 2r)}{2(a - r)r^3} \,,$$

$$\frac{\partial^2 D}{\partial r_1\, \partial r_2} = \frac{4a^2(a-r_1)(a-r_2)}{(-2r_1r_2 + a(r_1+r_2))^3} \rightarrow \frac{a^2}{2ar^3 - 2r^4}\,.$$

Note especially that

$$\frac{\partial^2 D}{\partial r_1^2} - \frac{\partial^2 D}{\partial r_1\, \partial r_2} \rightarrow -\frac{a}{r^2(a-r)}$$

diverges at the boundary, $r \rightarrow a$.

When $r' = r$, 1st and 2nd derivatives w.r.to θ_j are 0 when $u = 0$ (very complicated otherwise).

Derivatives of $\overline{T}$

For any smooth curved surface,

$$\frac{\partial \overline{T}}{\partial t} = -2tD(t^2 + 2\sigma)^{-2},$$

$$\frac{\partial^2 \overline{T}}{\partial t^2} = 8t^2 D(t^2 + 2\sigma)^{-3} - 2D(t^2 + 2\sigma)^{-2},$$

$$\frac{\partial \overline{T}}{\partial r} = -2D(t^2 + 2\sigma)^{-2}\,\frac{\partial \sigma}{\partial r} + \frac{\partial D}{\partial r}\,(t^2 + 2\sigma)^{-1},$$

$$\frac{\partial^2 \overline{T}}{\partial r^2} = 8D(t^2+2\sigma)^{-3}\left(\frac{\partial \sigma}{\partial r}\right)^2 - 2D(t^2+2\sigma)^{-2}\,\frac{\partial^2 \sigma}{\partial r^2}$$
$$- 4(t^2+2\sigma)^{-2}\,\frac{\partial D}{\partial r}\,\frac{\partial \sigma}{\partial r} + (t^2+2\sigma)^{-1}\,\frac{\partial^2 D}{\partial r^2}\,,$$

$$\frac{\partial^2 \overline{T}}{\partial r'\,\partial r} = 8D(t^2+2\sigma)^{-3}\frac{\partial \sigma}{\partial r'}\,\frac{\partial \sigma}{\partial r} - 2D(t^2+2\sigma)^{-2}\frac{\partial^2 \sigma}{\partial r'\,\partial r}$$
$$- 2(t^2+2\sigma)^{-2}\left(\frac{\partial D}{\partial r}\,\frac{\partial \sigma}{\partial r'} + \frac{\partial D}{\partial r'}\,\frac{\partial \sigma}{\partial r}\right)$$
$$+ (t^2+2\sigma)^{-1}\,\frac{\partial^2 D}{\partial r'\,\partial r}\,,$$

and similarly for θ derivatives.

For sphere, with $\mathbf{r}' = \mathbf{r}$, $\xi = \frac{1}{4}$,

$$D = \left(1 \pm \frac{l}{2a}\right)^{-1} = \frac{a}{r}\,, \qquad 2\sigma = 4(a-r)^2.$$

Energy:

$$\frac{\partial^2 \overline{T}}{\partial t^2} = \frac{8t^2 a}{r}(t^2 + 2\sigma)^{-3} - \frac{2a}{r}(t^2 + 2\sigma)^{-2}$$
$$= \frac{2a}{r}\,\frac{3t^2 - 4(a-r)^2}{[t^2 + 4(a-r)^2]^3}\,.$$

Tangential pressure:

$$\frac{1}{r}\,\frac{\partial \overline{T}}{\partial r} = 4(a-r)\,\frac{a}{r^2}(t^2+2\sigma)^{-2} - \frac{a}{2r^3}(t^2+2\sigma)^{-1}.$$

First term $= 0$ on sphere. Sign changes in exterior(?).

$$\frac{1}{r^2}\,\frac{\partial \overline{T}}{\partial \theta'\,\partial \theta} = 2\left(\frac{a}{r}\right)^2 (t^2+2\sigma)^{-2}.$$

$$p_\perp = -\left(\frac{a}{r}\right)(t^2+2\sigma)^{-2} - \frac{a}{8r^3}(t^2+2\sigma)^{-1}.$$

So far, ρ and $p_\perp$ are consistent with flat plate.

Radial pressure:

$$\frac{\partial^2 \overline{T}}{\partial r^2} = \frac{16a\sigma}{r}(t^2 + 2\sigma)^{-3} - \left(\frac{2a}{r} + \frac{2al}{r^2}\right)(t^2 + 2\sigma)^{-2}$$
$$+ \frac{a(a-2r)}{2(a-r)r^3}(t^2 + 2\sigma)^{-1};$$

$$\frac{\partial^2 \overline{T}}{\partial r' \, \partial r} = \text{(same top line)}$$
$$+ \frac{a^2}{2(a-r)r^3}(t^2 + 2\sigma)^{-1}.$$

All but last terms cancel in p_r; last terms diverge!

Why? Optical approximation and even higher-order DeWitt–Christensen expansions break down at boundary [McAvity & Osborn, CQG **8** ('91) 603].

Possible responses:

1. Get p_r from conservation law.
2. Use full multiple-reflection formula for $\overline{T}$.

Conservation law

$$\frac{\partial p_r}{\partial r} + \frac{2p_r}{r} - \frac{2}{r}\, p_\perp = 0.$$

$$
\begin{aligned}
p_r = & \frac{C}{r^2} \quad \text{(homogeneous soln.)} \\
& + \frac{a(a-r)}{t^2 r^2} \, \frac{1}{t^2 + 2\sigma} \\
& - \frac{2a^3}{t^3 r^2} \, \frac{4a^2 + 3t^2}{(4a^2 + t^2)^2} \, \tan^{-1}\left(\frac{t}{l}\right) \\
& + \left(\frac{a}{r}\right)^2 \frac{1}{(4a^2 + t^2)^2} \ln\left(\frac{t^2 + 2\sigma}{r^2}\right) \\
& + \frac{a}{4r^3} \, \frac{1}{4a^2 + t^2} \, .
\end{aligned}
$$

Third term $\to -\dfrac{\pi}{4at^3} \qquad (r = a,\ t \ll a)$;
correct pressure for surface energy, $E = \dfrac{\pi}{8t^3}$, and should change sign on outside.

But how do I know that $C = 0$? Can't look at $r = 0$ because optical approx. blows up there (caustic).

Also, E is independent of ξ, so $p_r(a)$ must be. How does that turn out? ($\xi = \frac{1}{6} \ \Rightarrow$ leading terms in ρ and $p_\perp$ vanish.)

Multiple-reflection expansion

[Balian & Bloch, Ann. Phys. …]

$$\overline{T}(\mathbf{r},\mathbf{r}',t) = \int_0^\infty \operatorname{Im} G(\mathbf{r},\mathbf{r}',k) e^{-kt}\, dk.$$

Change notation: r_1 and r_2 $\equiv$ distances of $\mathbf{r}$ and $\mathbf{r}'$ from integration variable $\mathbf{r}_*$ on the sphere; $l = r_1 + r_2$.

$$\operatorname{Im} G_{(1 \text{ refl.})} \propto \operatorname{Im} \frac{\partial}{\partial r_1} \left(\frac{e^{ikr_1}}{r_1} \right) \frac{e^{ikr_2}}{r_2}$$

$$= -\frac{\sin k(r_1+r_2)}{r_1^2 r_2} + \frac{k \cos k(r_1+r_2)}{r_1 r_2} .$$

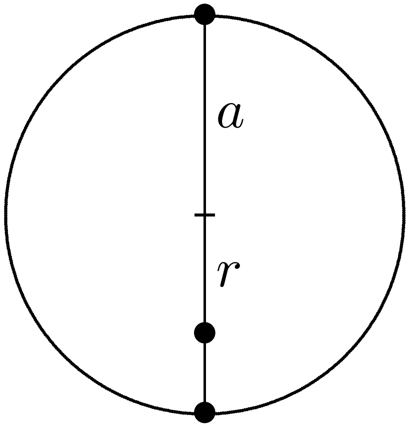
So single-reflection term is

$$\overline{T} \propto \int dS \, \cos\phi \, \left[\frac{1}{r_1 r_2} \, \frac{l^2 - t^2}{(l^2 + t^2)^2} + \frac{1}{r_1^2 r_2} \, \frac{l}{l^2 + t^2} \right],$$

$\phi =$ angle between $\mathbf{r} - \mathbf{r}_*$ and $\hat{\mathbf{n}}$.
When $\mathbf{r}' = \mathbf{r}$, $r \equiv |\mathbf{r}_1|$, $z \equiv l = 2r_1$, after geometry

$$\overline{T} \propto -4 \int dz \, \frac{1}{r z^2} \left(\frac{z^2}{4} + a^2 - r^2 \right) \frac{3z^2 + t^2}{(z^2 + t^2)^2}$$

$$= \frac{z^2 + 4(a^2 - r^2)}{r(t^2 + z^2)} - \frac{2}{rt} \, \tan^{-1} \left(\frac{z}{t} \right) \Bigg|_{l_{\min}=2|a-r|}^{l_{\max}=2(a+r)} .$$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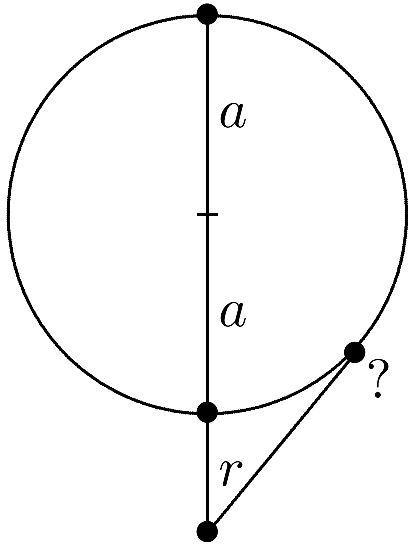
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Lower limit's contribution:

$$\overline{T} \propto -2\,\frac{(a-r)^2+(a^2-r^2)}{r|a-r|(t^2+4(a-r)^2)} + \frac{2}{rt}\,\tan^{-1}\left(\frac{l_{\min}}{t}\right).$$

First term is the optical approximation:

$$\overline{T} \propto \frac{a}{r}\,\frac{1}{t^2+2\sigma} = \frac{D}{t^2+2\sigma}\,.$$

But

- Sign discrepancy between interior and exterior?
- Upper limit's term should have caustic sign?
- Regularity at $\mathbf{r}=0$?
- More reflections irrelevant?