Quantum Chaos: Electron Waves in Nanostructures and Freak Waves in the Ocean

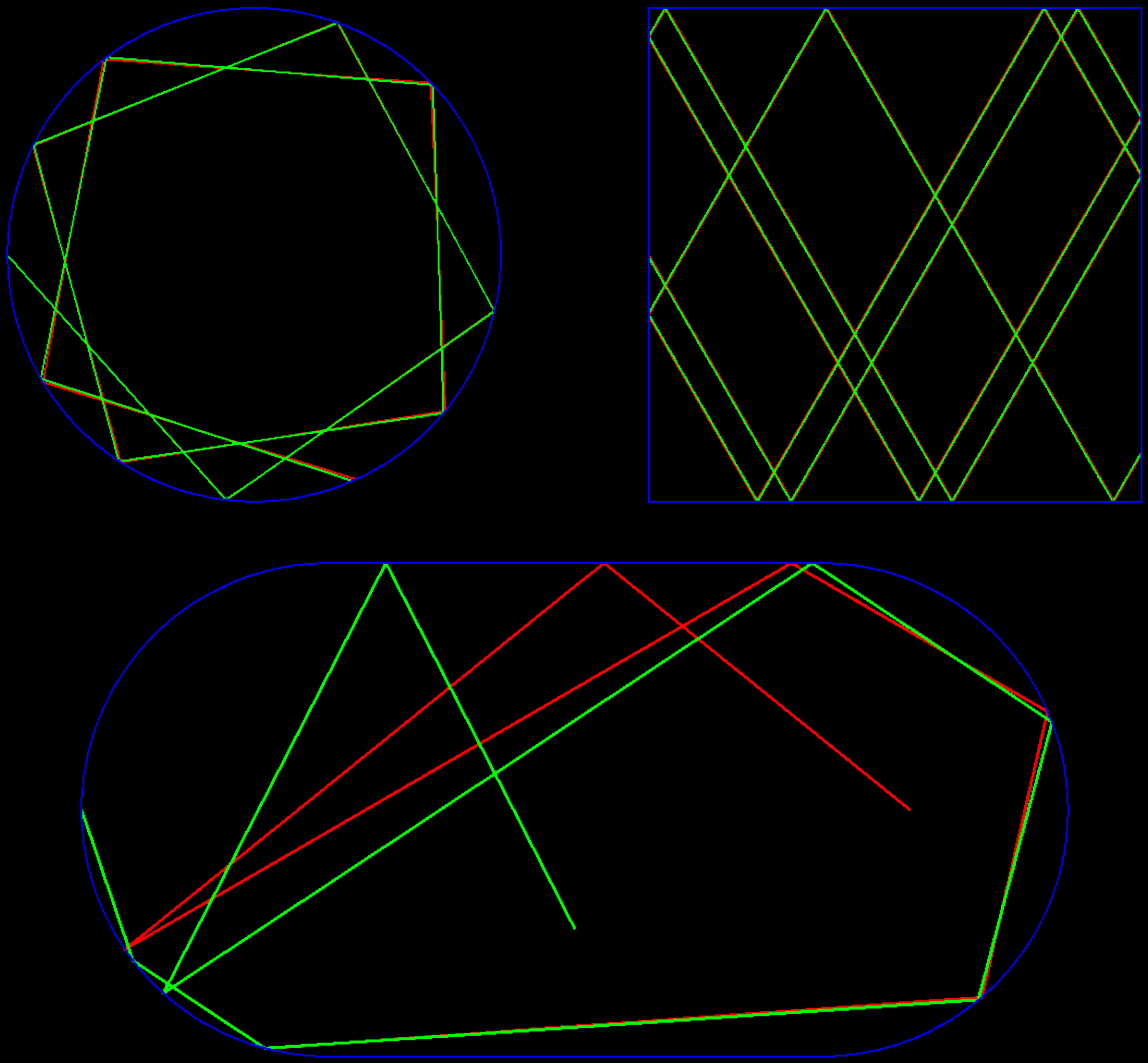
Lev Kaplan
Tulane University

Talk outline:

- Chaos: what is it and why should we care?
  - Classical (ray) chaos
  - Quantum (wave) chaos
- Two applications: motion in random potential
  - Electron flow in nanostructures (10^{-7} m)
  - Freak waves on the ocean (10^{5} m)

Classical (ray) chaos

- *Regular* system: perturbation in initial conditions grows at most linearly with t
  - One-dimensional or separable motion
  - Fully regular behavior unusual in $d \geq 2$
- Most systems are *chaotic*
  - Perturbations generally grow exponentially with time: $\Delta x(t) \sim e^{\alpha t}\, \Delta x(0)$
  - Unpredictable determinism! (no information produced by dynamics)



Quantum (wave) chaos

- Replace particle (or ray) bouncing in a box with analogous wave system: vibrating drumhead
- Wave system *not* chaotic
  - Exponential sensitivity to infinitesimal change in initial conditions washed out by finite wavelength (uncertainty principle)

$$\int \psi^*(x,0)\psi(x,0)dx = 0.9 \quad \Rightarrow \quad \int \psi^*(x,t)\psi(x,t)dx = 0.9$$

- But, correspondence principle (short wavelengths)

Quantum (wave) chaos

- Definition: “Study of quantum (or classical wave) systems whose classical (or ray) limit is chaotic”
- Look at spectral, wave function, transport properties
- In general, no analytic solutions
- One approach: brute-force numerical calculation
  - Little insight, need to re-do for new parameters
- Instead, we can
  - Search for “universal” statistical predictions, valid for all chaotic systems (RMT or random waves)
  - Look for specific correspondence between classical and quantum properties

Quantum (wave) chaos

- Key tool: semi-classical evolution
  - Sum over all paths (Feynman) approximated by sum over classical paths with phases
  - Includes interference (double-slit) but not “hard quantum” effects such as tunneling, diffraction
  - Bridge between QM and our classical intuition

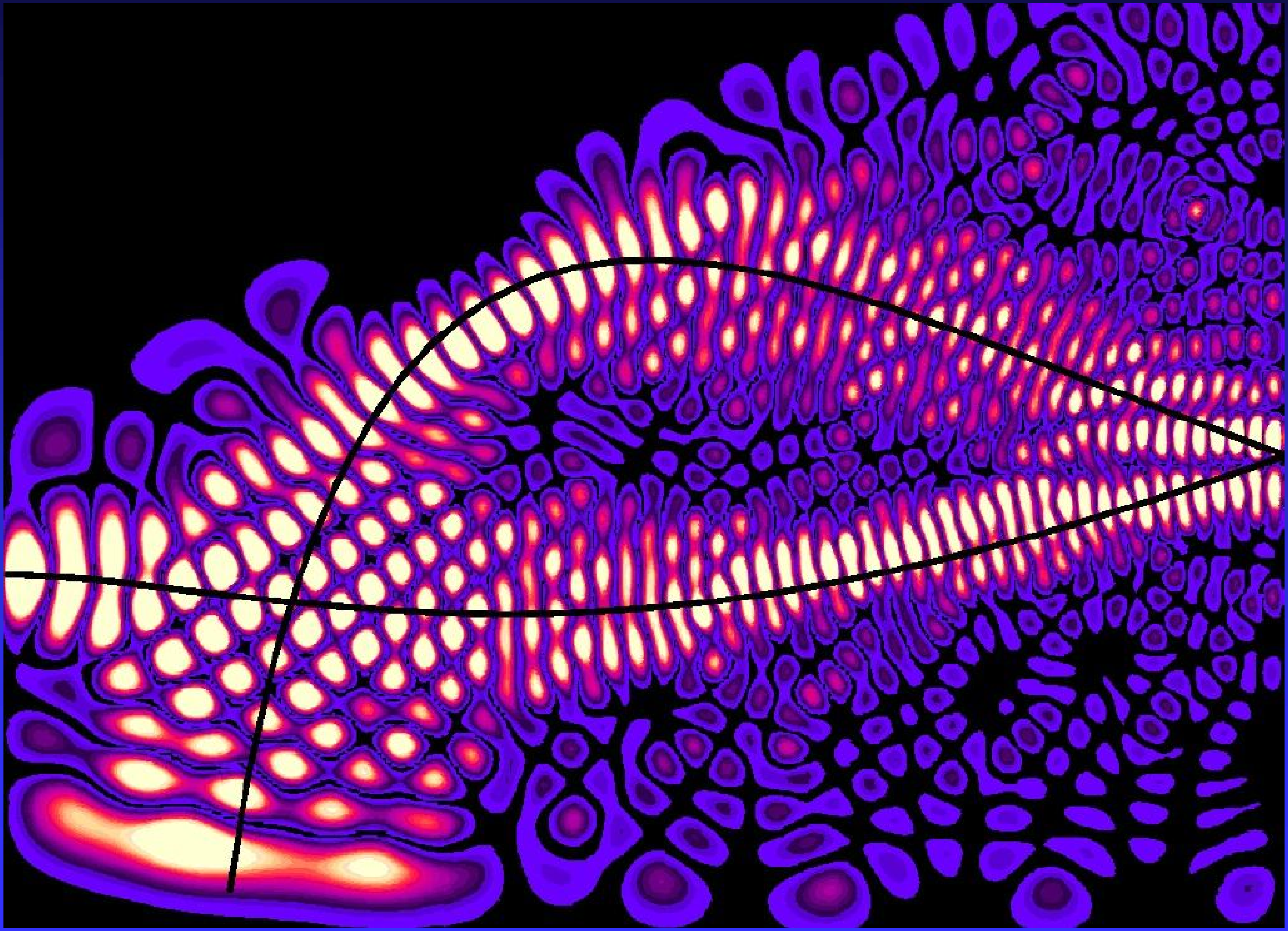
- Bring together insights, methods, examples from AMO, nuclear, nanostructures, microwaves, acoustics, mathematical physics, …

Quantum (wave) chaos: examples

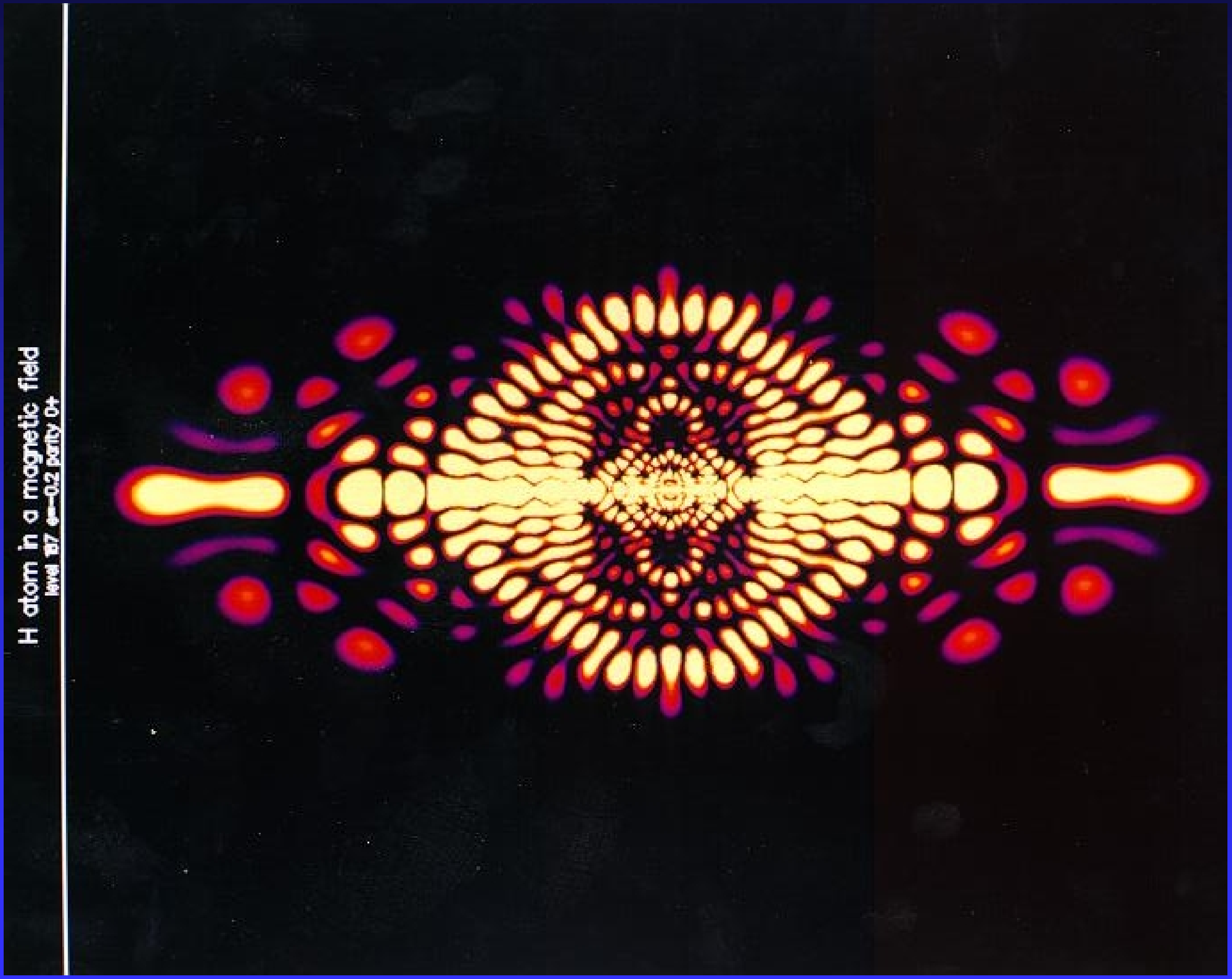
- Quantum:
  - Conductance through nanodevices
  - Hydrogen in strong magnetic field
  - Highly excited molecules
  - Quantum corrals

- Classical waves:
  - Microwave resonators
  - Acoustics: long-range sound propagation in ocean
  - Optics: directed emission from microlasers

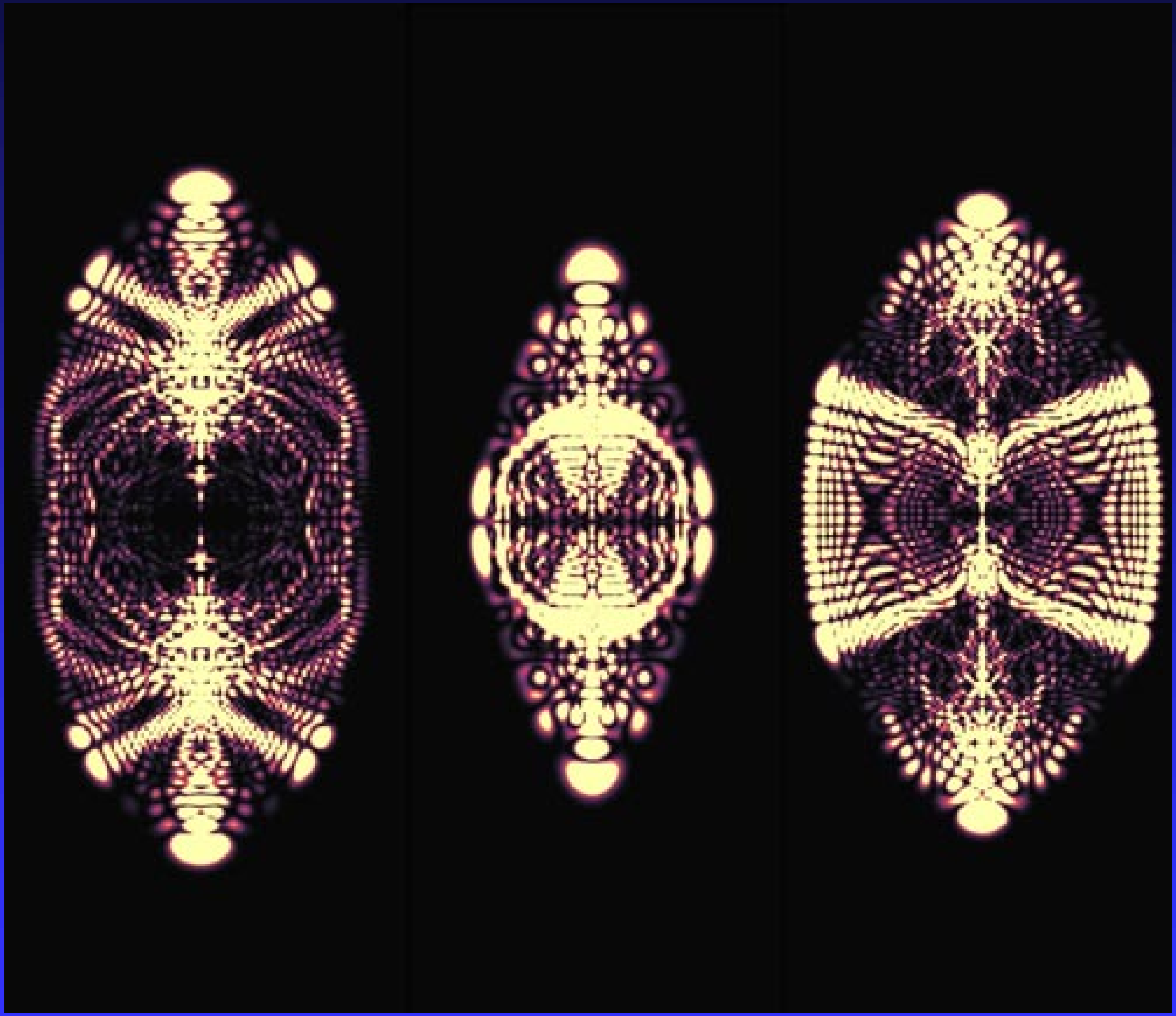
Conductance through tunneling diode (Monteiro et al, Nature 1997)



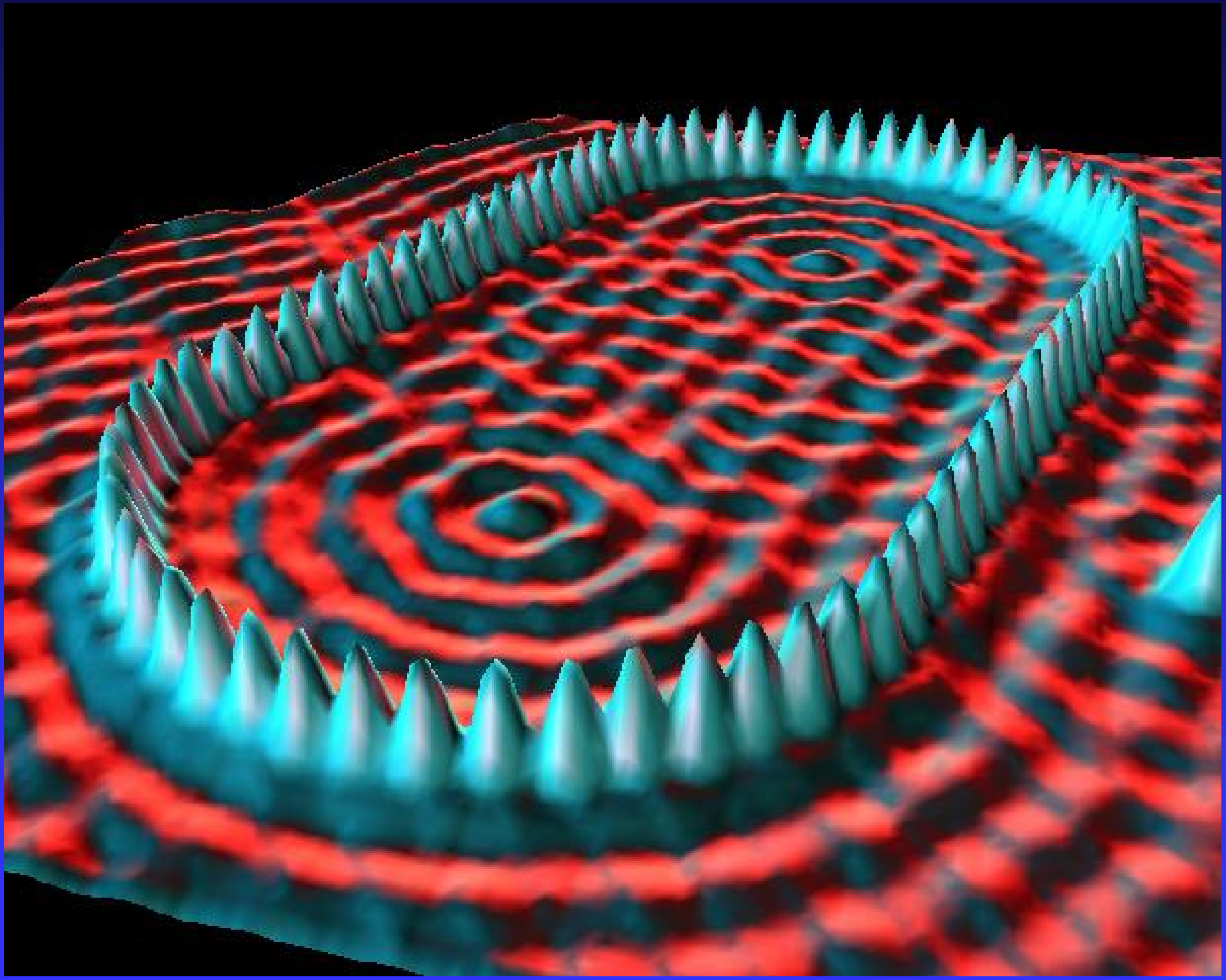
Hydrogen atom wave function in strong B field



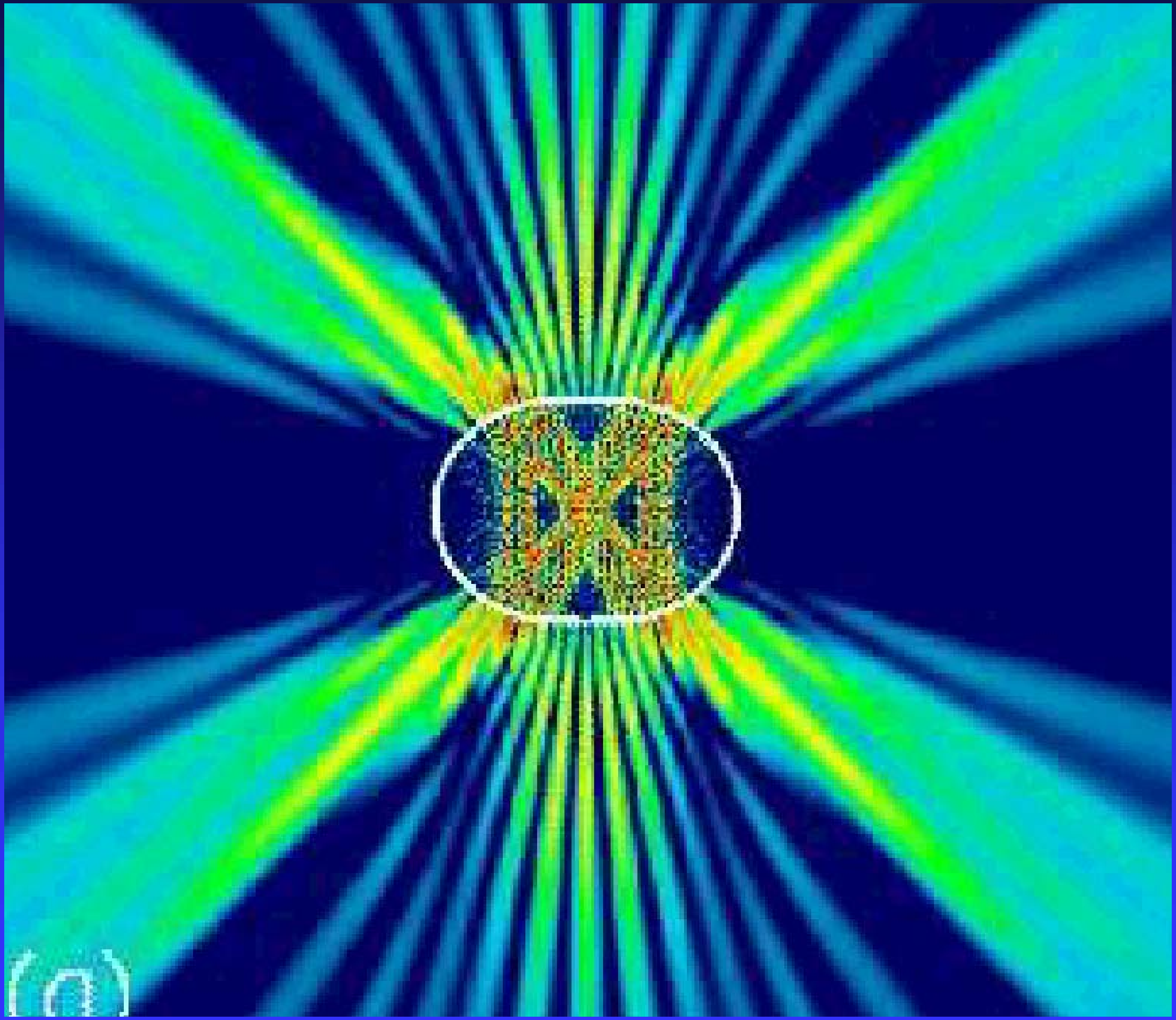
Wave functions for highly excited H_2 molecule



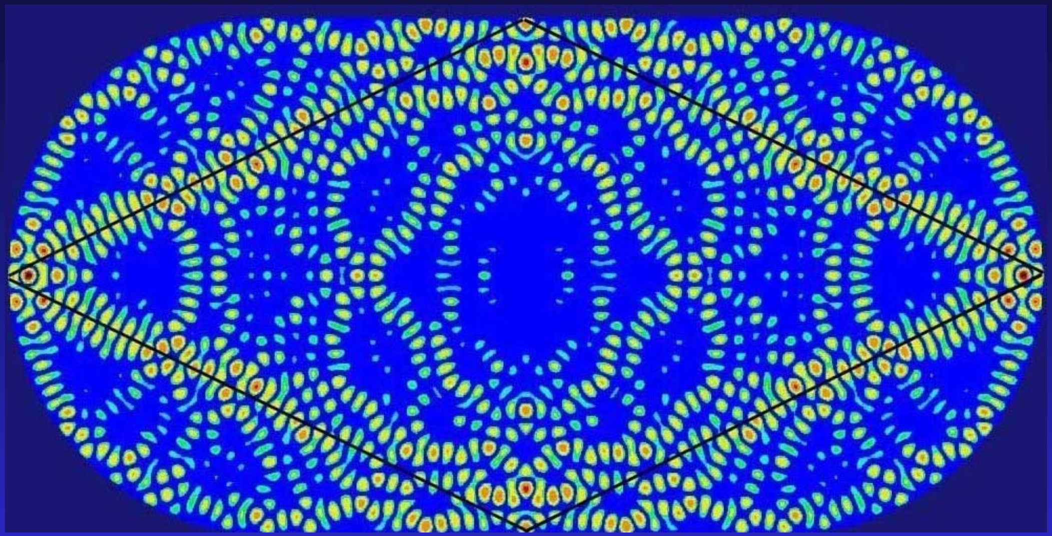
"Quantum corral": STM image of surface electron density (Crommie et al, Science 1993)

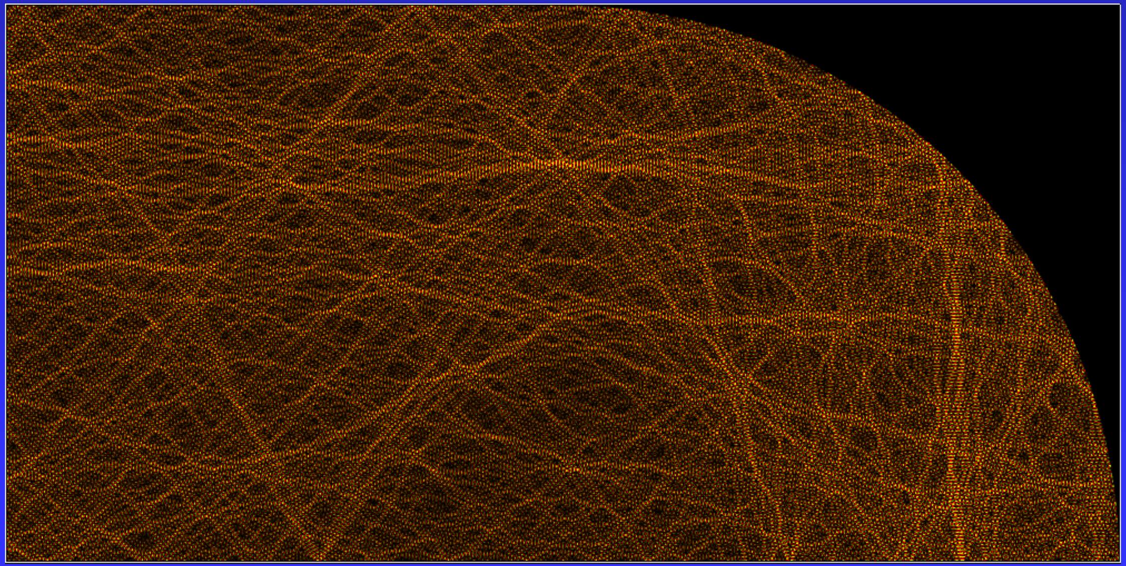


Directional emission from microlaser with dielectric resonator (Gmachl et al, Science 1998)



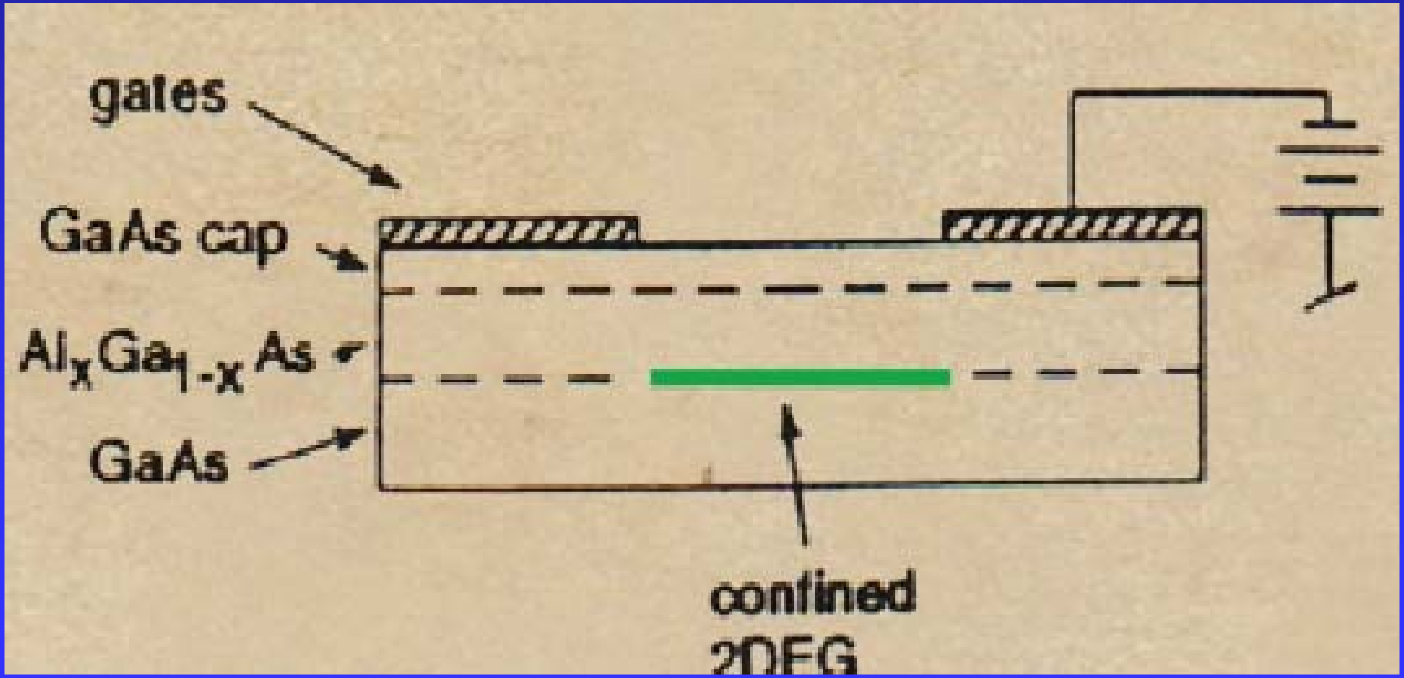
Numerical calculations for stadium billiard





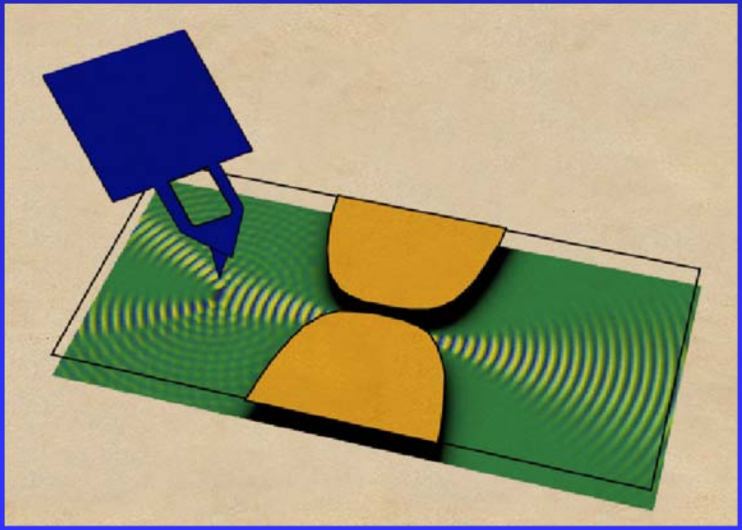
Application I: Electron Flow in Nanostructures

- Electrons confined to 2-dimensional electron gas (2DEG) inside GaAs-AlGaAs heterostructure
- Gate voltages used to create barriers inside 2DEG and carve out region through which electrons may flow



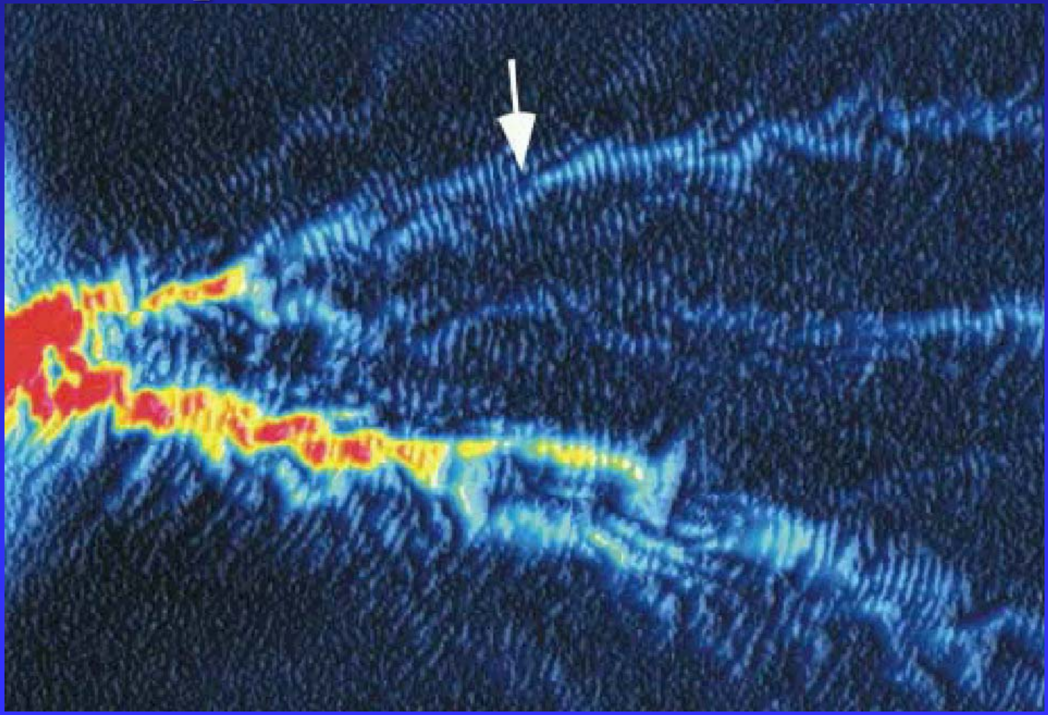
Application I: Electron Flow in Nanostructures

- Barriers chosen so electrons must pass through narrow quantum point contact (QPC)
- Scanning probe microscope technique used to image electron flow through such a 2DEG device
  - Negatively charged tip reflects current and reduces conductance through device
  - By measuring reduction in conductance as function of tip position, can map out regions of high and low current



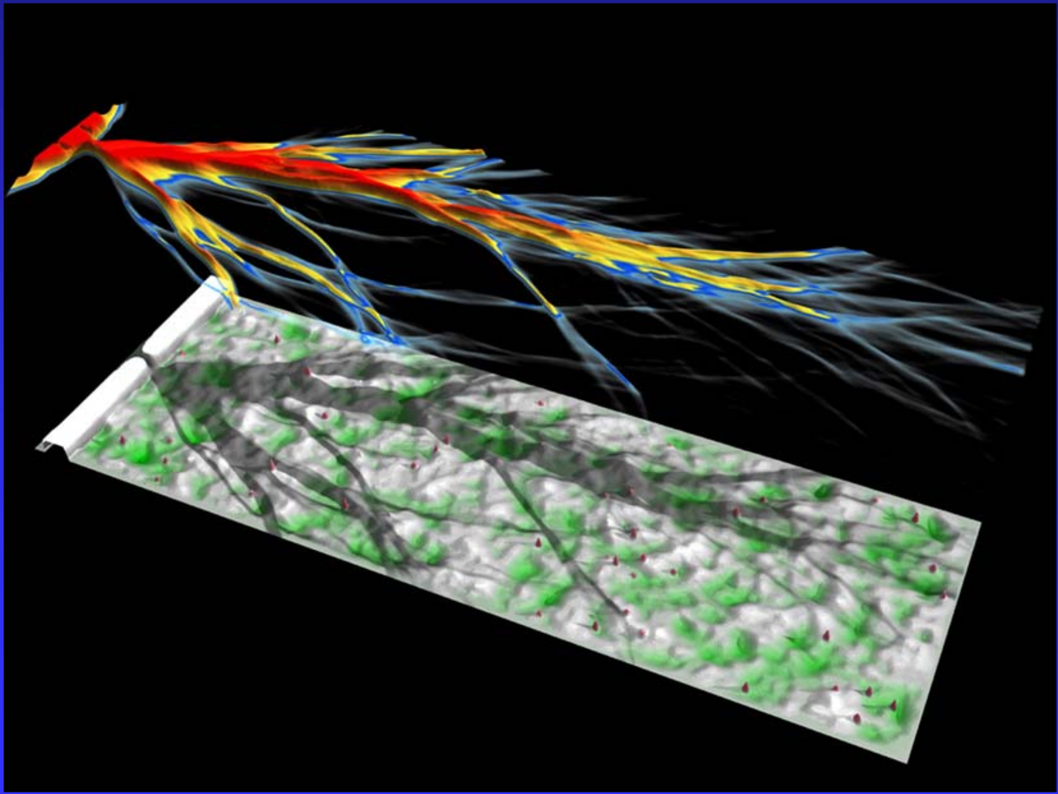
Application I: Electron Flow in Nanostructures

- Known: potential V(x,y) away from gates is not zero, but is random due to donor ions and impurities in 3d bulk
- Expected: outgoing flow should exhibit random wave pattern (Gaussian random amplitude fluctuations)
- Instead: observe strong current concentrated in small number of “branches” (Topinka et al, Nature 2001)



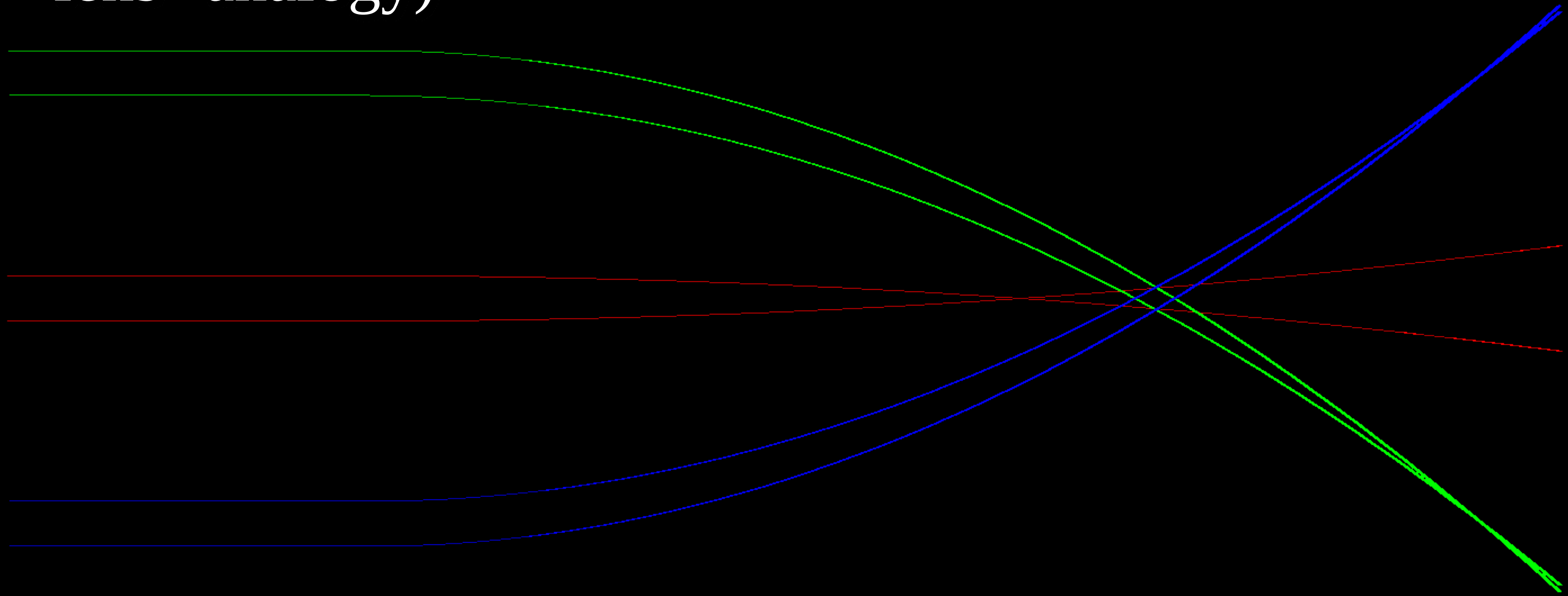
Understanding branching of electron flow

- Simplest description of random potential: Gaussian random with rms height V_{RMS} and correlation distance d
- Numerical simulations of the Schrödinger equation show qualitatively similar branching behavior
- Also seen in classical simulation!
  - Semiclassical or ray picture must be applicable
  - Branches do not correspond to "valleys" of the potential



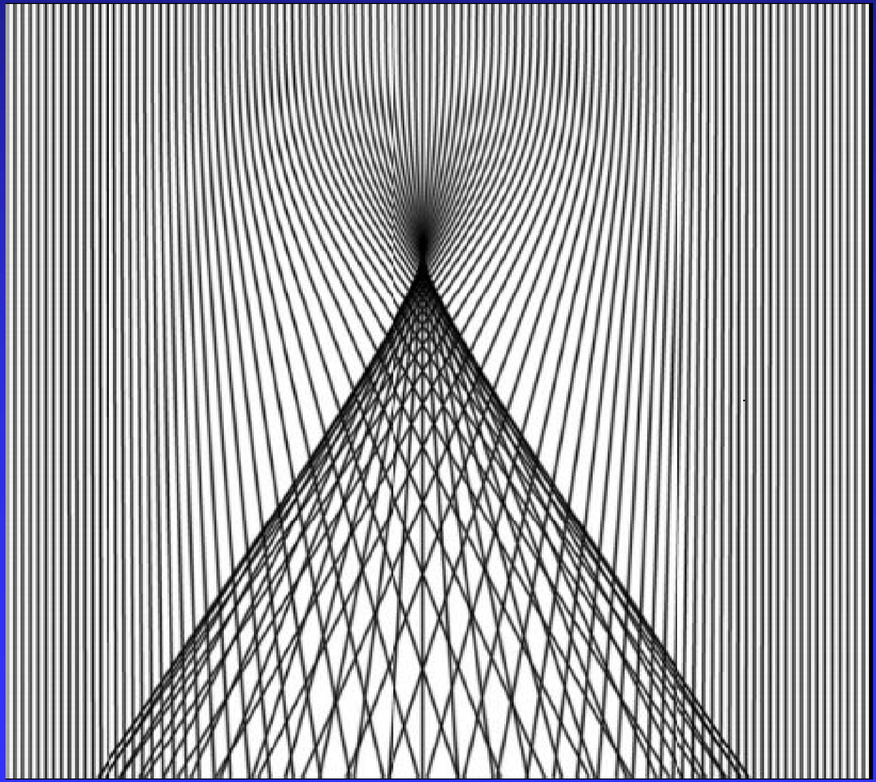
Understanding branching of electron flow

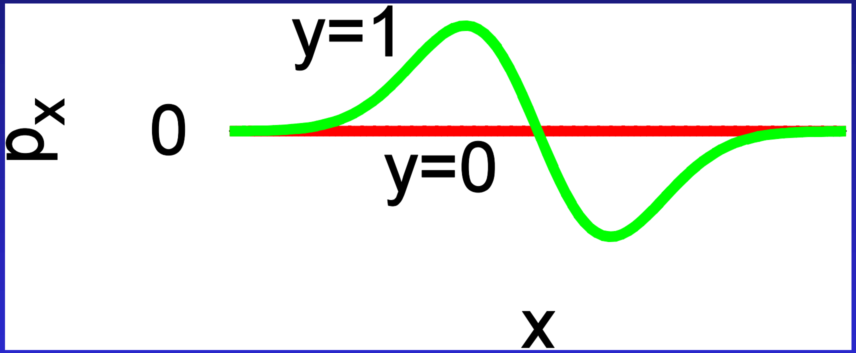
- Consider parallel incoming paths encountering single shallow dip in potential V(x,y)
- Focusing when all paths in a given neighborhood coalesce at a single point (caustic), producing infinite ray density
- Different groups of paths coalesce a different points ("bad lens" analogy)

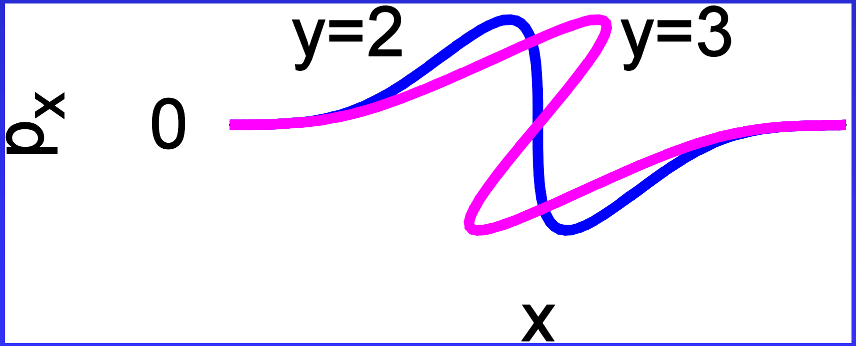


Understanding branching of electron flow

- Generic result: "cusp" singularity followed by two lines of "fold" caustics
- At each y after cusp singularity, we have infinite density at *some* values of x







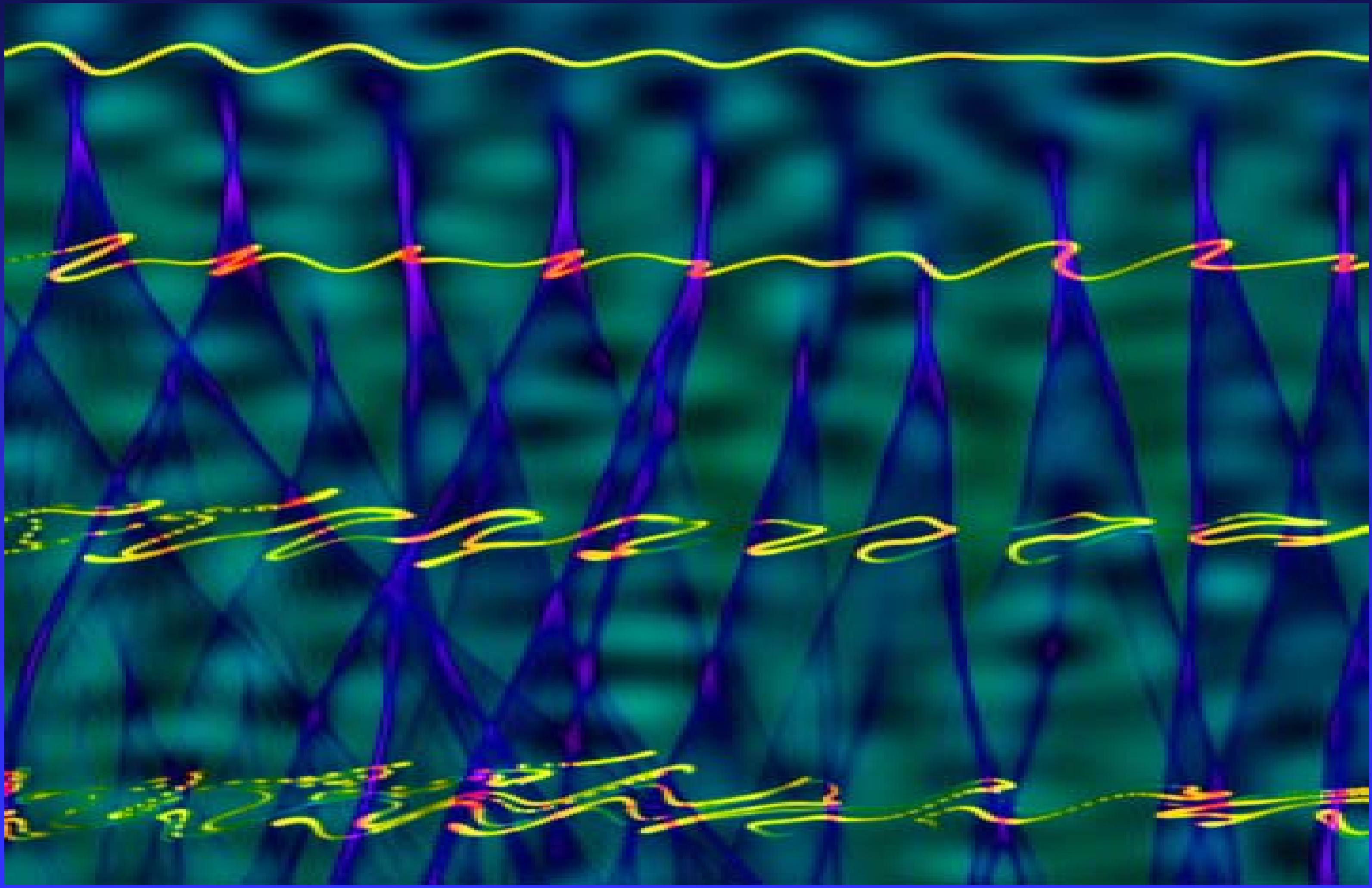
Phase space picture

Understanding branching of electron flow

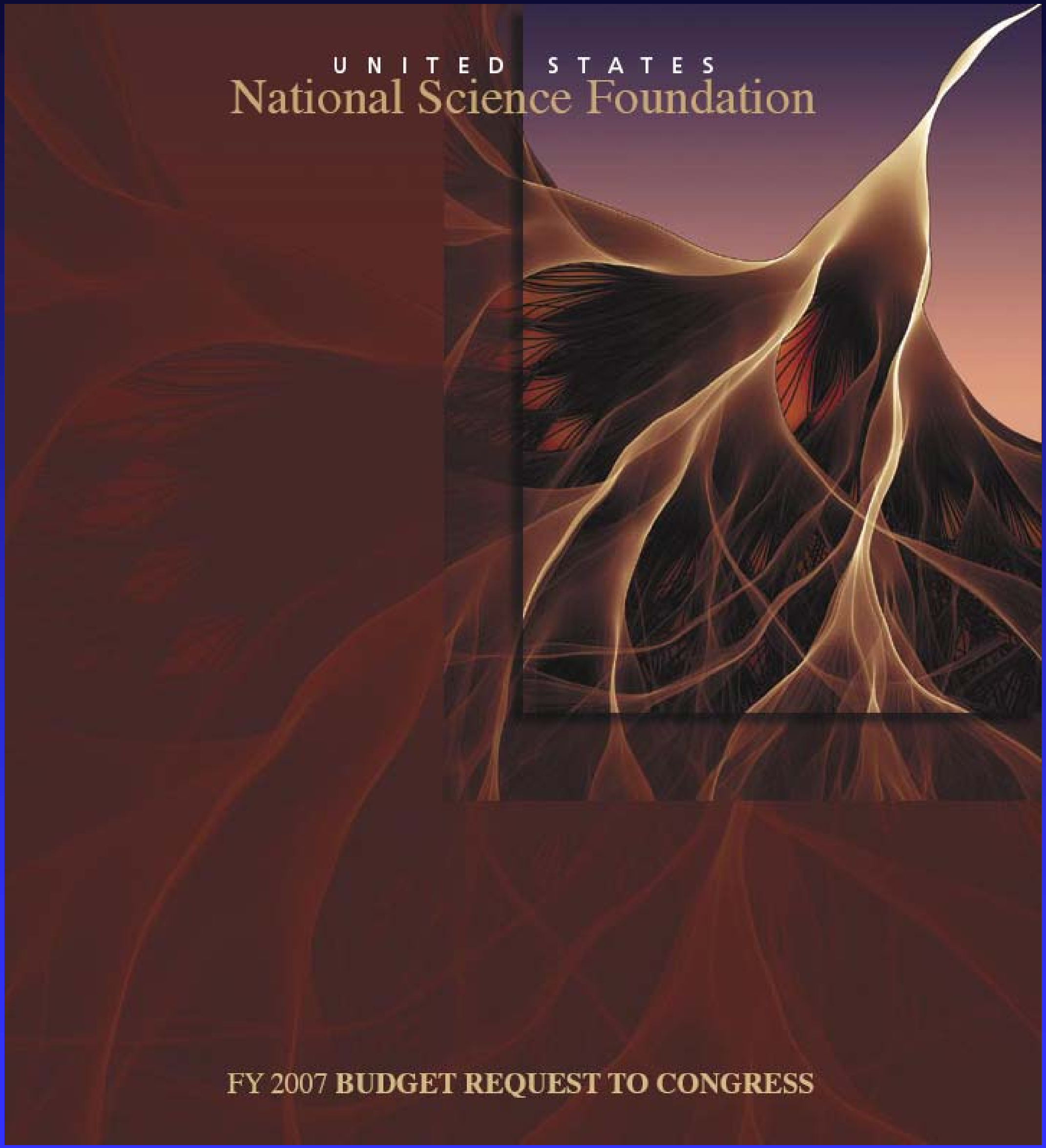
- Realistic situation: weak potential V_{RMS} << KE => small angle scattering
- Single bump or dip of size ~d insufficient to produce cusp singularity
- Instead, first singularities formed after typical distance scale $D \sim d\,(KE/V_{RMS})^{2/3} >> d$
- Further evolution: exponential proliferation of caustics
  - Tendrils decorate original branches
  - Universal branch statistics with single distance scale *D*
- Individual branch locations & heights depend on fine details of random potential, but *statistics* depend only on dimensionless parameter KE/V_{RMS}

Understanding branching of electron flow

- Multiple branching



More artistic visualization of electron flow



(Eric Heller)

Computing statistics of branched flow

- Wave mechanics: caustic singularity washed out on wavelength scale (uncertainty principle)
  - Visible branch only if smeared intensity above background
- At long distances y >> D from QPC:
  - Number of caustics grows exponentially
  - Typical intensity of each caustic decays exponentially due to stretching of phase space manifold
- But not all pieces of manifold stretch at same rate
  - Visible branch occurs only when singularity occurs in piece of manifold that had stretched anomalously little

Computing statistics of branched flow

- When all distances expressed in terms of D, everything governed by 2 dimensionless numbers (describing log-normal distribution of stretching factors)
  - α: average rate of stretching
  - β: variance of stretching rate
- Then # of branches decays exponentially as $\exp(- (\alpha^2 / \beta) \; y / D)$
- Intensity of strongest branch:

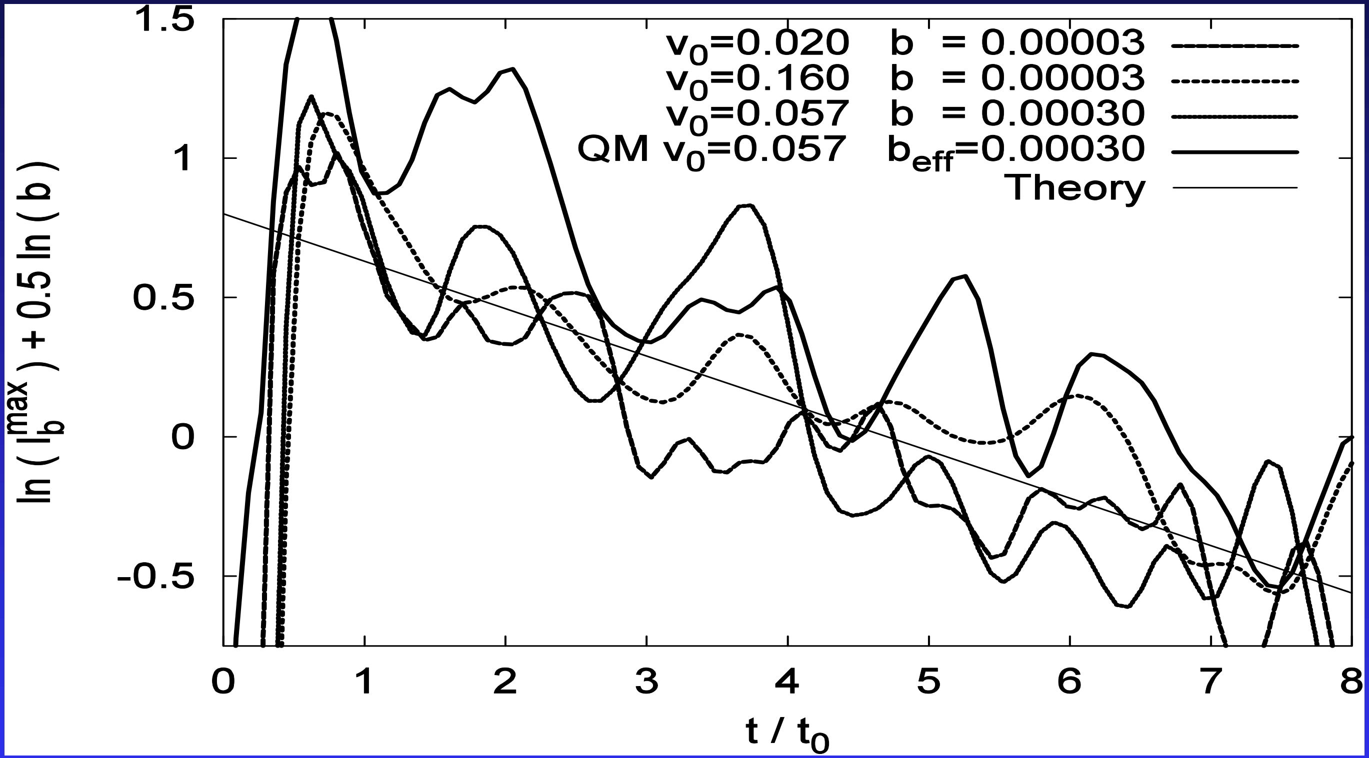
$$\ln (I^{max}) = (1/2) \ln (d / \lambda) - \gamma (y / D)$$

where $\gamma = \alpha - (\beta /2) [\mathrm{sqrt}(1+4 \alpha / \beta) - 1]$

and λ is the wavelength

Numerical simulations

- Intensity of strongest branch:



- Similarly can compute distribution of branch heights, fraction of space covered by branches, etc.

Application II: Freak Waves

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- ~ 10 large ships lost per year to presumed rogue waves – usually no communication
- Also major risk for oil platforms in North Sea, etc.
- Probability seems to greatly exceed random wave model predictions
- Large rogue waves have height of 30 m or more, last for minutes or hours
- Long disbelieved by oceanographers, first hard evidence in 1995 (North Sea)
- Tend to form in regions of strong current: Agulhas, Kuroshio, Gulf Stream

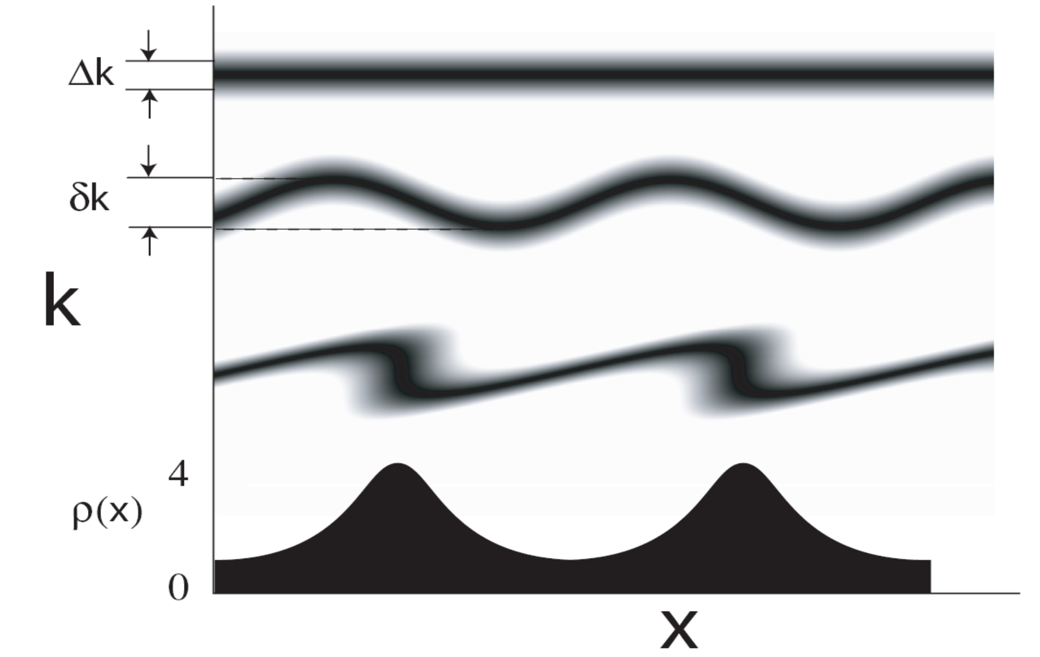
- Various explanations: nonlinear instabilities …
- Here, focus on refraction of incoming wave (velocity v) by random current eddies (typical current speed $u_{RMS} << v$)
- For deep water surface gravity waves with current

$$\omega(\vec{r},\vec{k}) = \sqrt{gk} + \vec{u}(\vec{r})\cdot\vec{k} \qquad \frac{dk_x}{dt} = -\frac{\partial\omega}{\partial x} \qquad \frac{dx}{dt} = \frac{\partial\omega}{\partial k_x}$$

- Different dispersion relation:
  - Electron waves: $E \sim p^2 \Rightarrow \omega \sim k^2 \Rightarrow v \sim k$
  - Surface water waves: $\omega \sim k^{1/2} \Rightarrow v \sim k^{-1/2}$

Calculation: Freak Waves

- Also, initial spread of wave directions important
  - New parameter $\Delta\theta$
- Causes smearing of singularities, on scales > wavelength
- “Hot spots” of enhanced average energy density remain as reminders of where caustics would have been

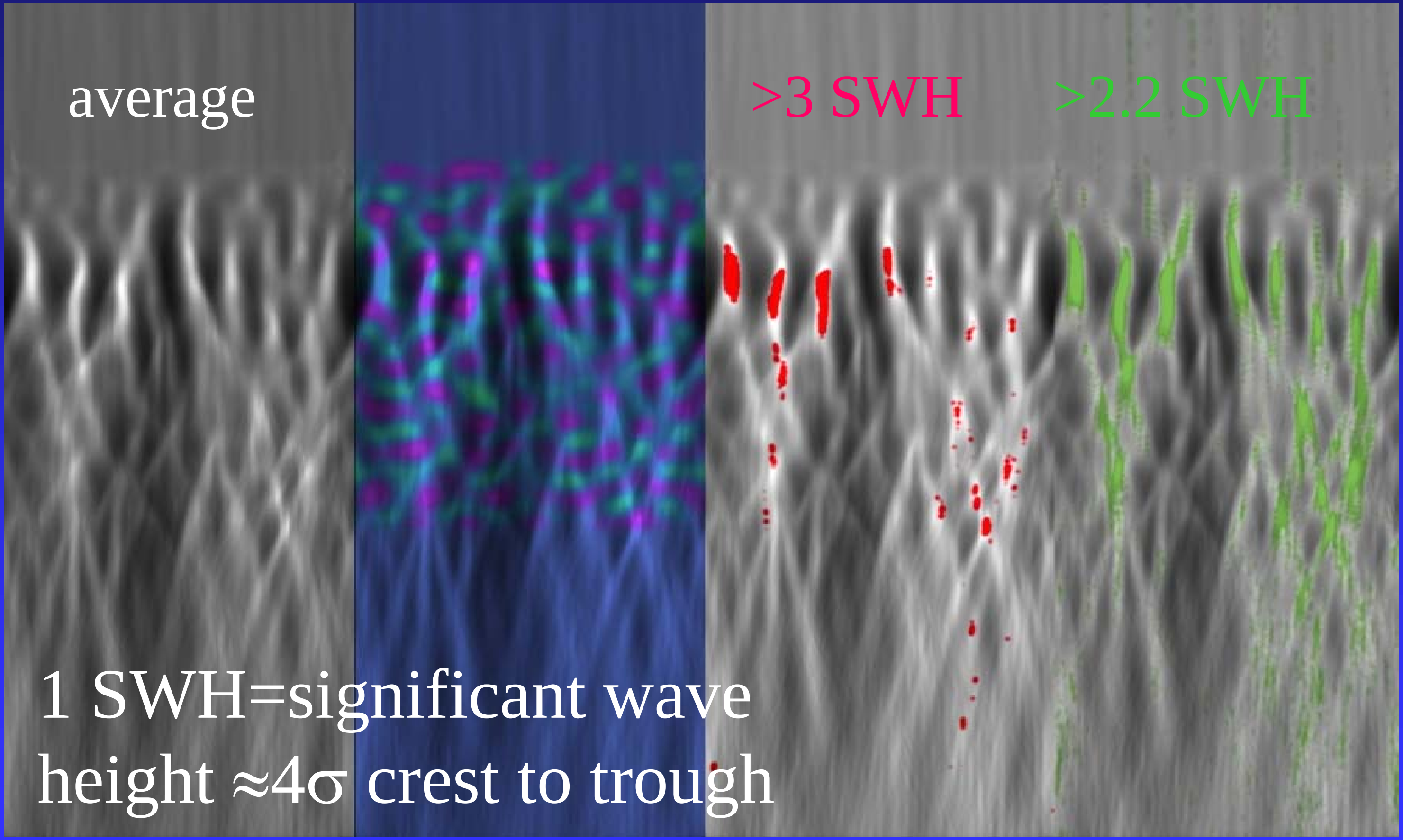


Calculation: Freak Waves

- Wave height intensity distribution: $P(h) = \int P_\sigma(h) f(\sigma) d\sigma$
- Obtained by superposing locally Gaussian wave statistics on pattern of "hot/cold spots" caused by refraction
- $\sigma^2(\vec{r})$ is position-dependent variance of the water elevation (high in focusing regions, low in defocusing regions)
- Probability f(σ) can be computed if "freak index" $\gamma = \Delta\theta \ (v/u_{RMS})^{2/3}$ is known
- $P_\sigma(h) = (h/\sigma^2)\exp(-h^2/2\sigma^2)$ Rayleigh distribution

Implications for Freak Wave Statistics

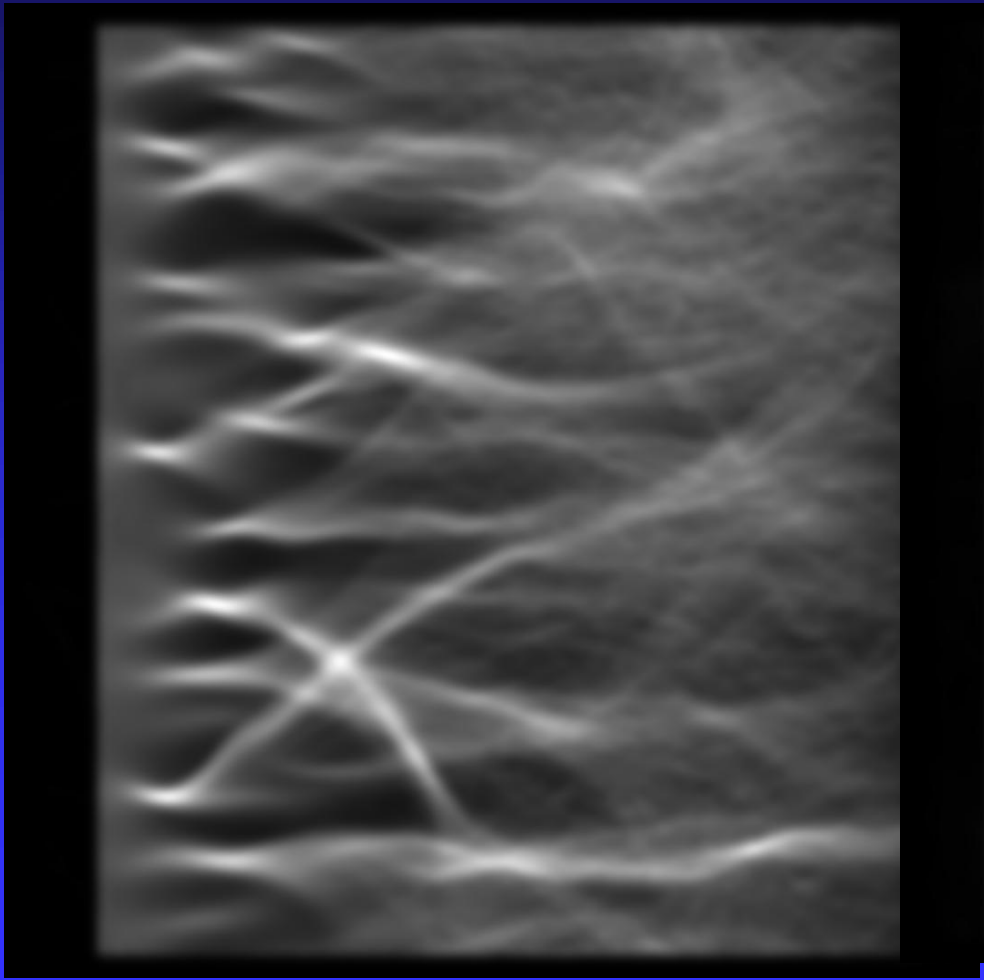
- Simulations (using Schrodinger equation): long-time average and regions of extreme events



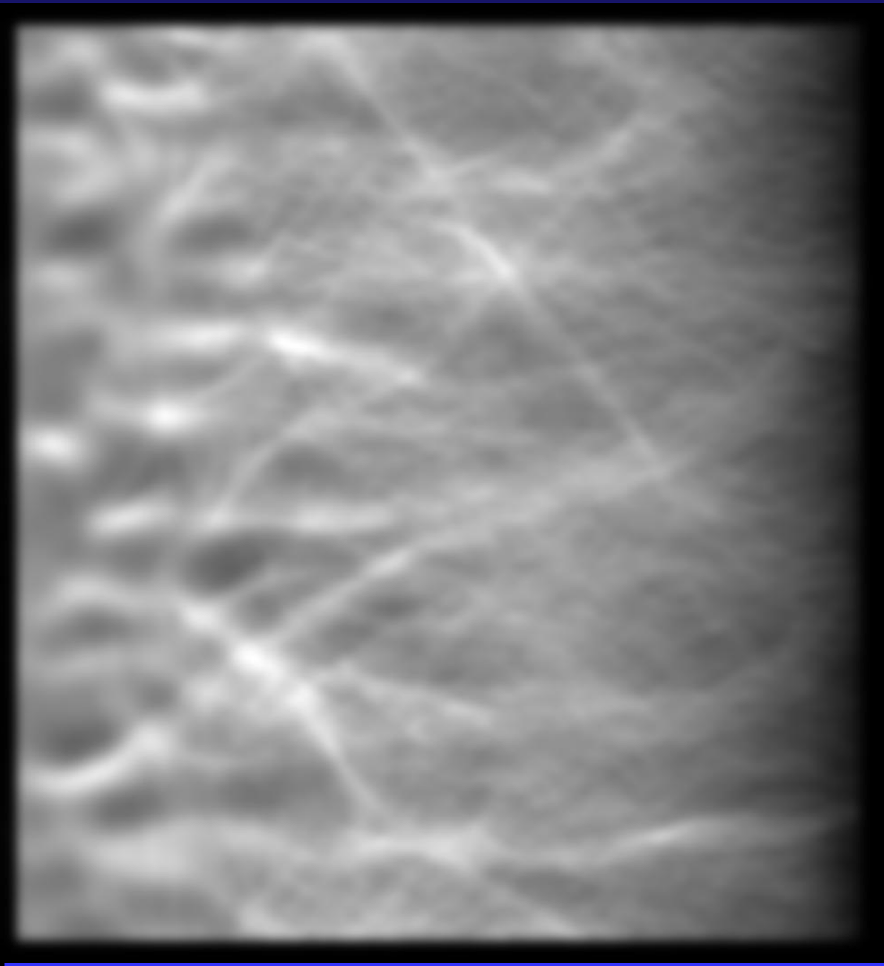
$\gamma \approx 2$

Typical ray calculation for ocean waves

$\Delta\theta = 5^{o}$



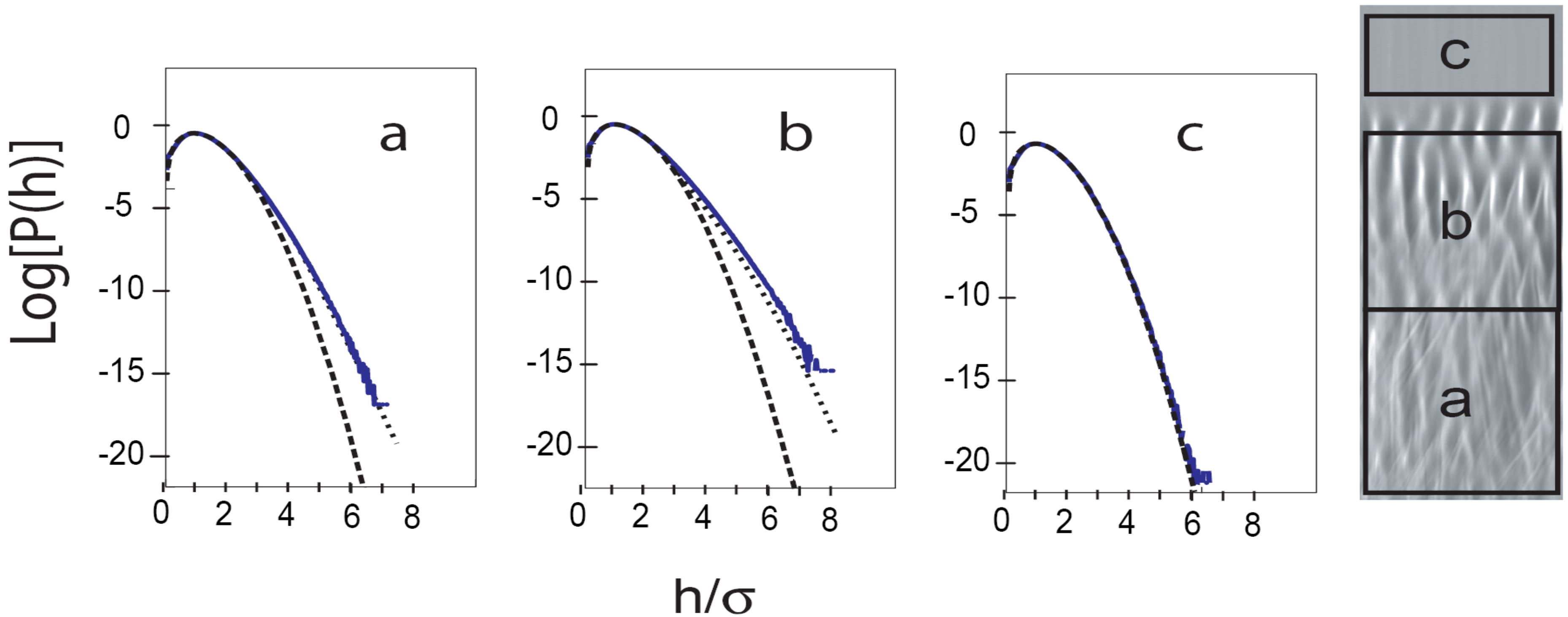
$\Delta\theta = 25^{o}$



Implications for Freak Wave Statistics

- Modified distribution of wave heights

$\gamma = 3.4$



Dashed = Rayleigh

Dotted = Theory based on *locally* Gaussian fluctuations

Note: significant deviations from Rayleigh in extreme tail

Analytics: limit of small freak index

- Rayleigh: $P(h > x \bullet SWH) = \exp(-2x^2/\sigma^2)$
- Average: $P(h > x \bullet SWH) = \int \exp(-2x^2/\sigma^2) f(\sigma) d\sigma$
- γ<<1: $g(\sigma)$ Gaussian with mean 1 and small width $\delta \sim \gamma \sim 1/\Delta\theta$
- Stationary phase:

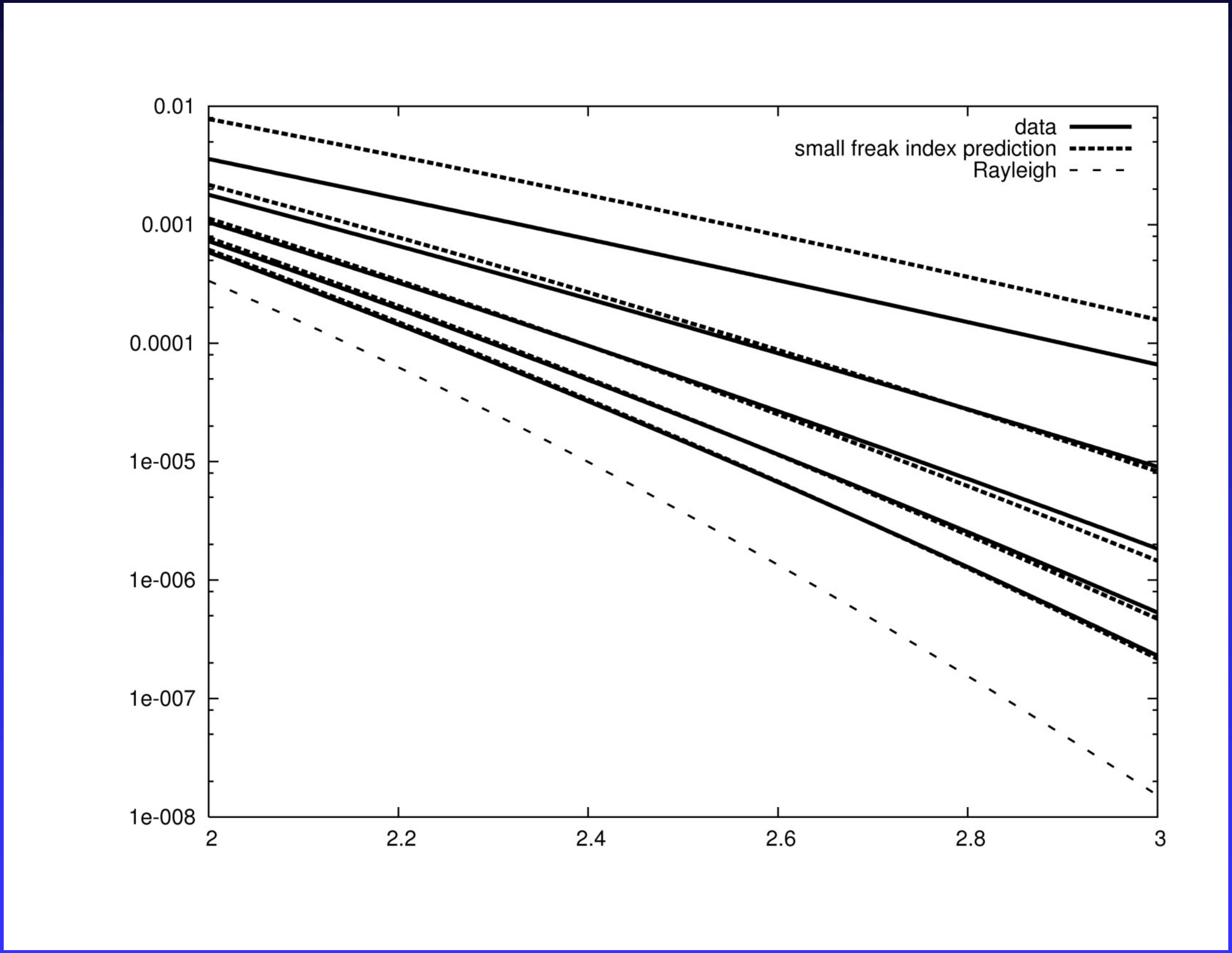
$$P(h > x \bullet SWH) = \sqrt{\frac{1+\varepsilon}{1+3\varepsilon}} \exp[-\varepsilon(1+\frac{3}{2}\varepsilon)/\delta^2]$$

where $\varepsilon(1+\varepsilon)^2 = 2\delta^2 x^2$

- Perturbative expansion: for $\varepsilon << 1$

$$P(h > x \bullet SWH) = [1+2\delta^2(x^4 - x^2)] \exp(-2x^2)$$

Wave height distribution for ocean waves



Summary:

- Quantum chaos: study of "generic" quantum (or classical wave) systems (i.e. lacking symmetries that make problem "trivial")
- Essential tools:
  - Semiclassical methods (ray-wave correspondence)
  - Statistical approaches
- Applications:
  - Statistics of branched electron flow in 2DEG
  - Probability of freak wave encounters on the ocean