I. 3D PISTONS WITH MIXED BOUNDARY CONDITIONS

August 21st By Zhonghai Liu

II. 3D RECTANGULAR PISTONS WITH MIXED BOUNDARY CONDITIONS——SCALAR FIELD

Lemma 1: Define two operators N and D as below

$$N_x^a f(x, x') = f(2a - x, x') \tag{1}$$

$$D_x^a f(x, x') = -f(2a - x, x') \tag{2}$$

then based on method of image we have (by number of reflections):

$$f^N(x, x') = f+(N_x^a f+N_x^b f)+(N_x^a N_x^b f+N_x^b N_x^a f)+(N_x^a N_x^b N_x^a f+N_x^b N_x^a N_x^b f)+\ldots \tag{3}$$

satisfies Neumann boundary conditions at both $x = a$ and $x = b$——**NN**;

$$f^D(x, x') = f+(D_x^a f+D_x^b f)+(D_x^a D_x^b f+D_x^b D_x^a f)+(D_x^a D_x^b D_x^a f+D_x^b D_x^a D_x^b f)+\ldots \tag{4}$$

satisfies Dirichlet boundary conditions at both $x = a$ and $x = b$——**DD**;

$$f^M(x, x') = f+(D_x^a f+N_x^b f)+(D_x^a N_x^b f+N_x^b D_x^a f)+(D_x^a N_x^b D_x^a f+N_x^b D_x^a N_x^b f)+\ldots \tag{5}$$

satisfies mixed boundary conditions: Dirichlet at $x = a$ and Neumann at $x = b$ or Neumann at $x = a$ and Dirichlet at $x = b$——**DN** or **ND**.

III. CYLINDER KERNEL OF 3D RECTANGULAR CAVITY WITH MIXED BOUNDARY CONDITIONS

The cylinder kernel in free space without any boundary is:

$$\overline{T}(t, \mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi^2}\frac{1}{(t^2 + |\mathbf{r} - \mathbf{r}'|)^2} \tag{6}$$

For the cavity with boundary condition $(NN)(NN)(DN)$, cylinder kernel can be written as:

$$\begin{aligned} \overline{T}^{NNM} =& \overline{T} + (N_x^0\overline{T} + N_x^a\overline{T} + N_y^0\overline{T} + N_y^b\overline{T} + D_z^0\overline{T} + N_z^c\overline{T}) \\ &+ (N_x^0 N_x^a\overline{T} + N_x^a N_x^0\overline{T} + N_y^0 N_y^b\overline{T} + N_y^b N_y^0\overline{T} + D_z^0 N_z^c\overline{T} + N_z^c D_z^0\overline{T}) + \dots \end{aligned} \tag{7}$$

From the point of view of classical paths, all paths above fall into 4 kinds:

Periodic paths—undergoing even number reflections at the walls;

Side paths——undergoing odd number reflections at the walls;

Edge paths——paths involving reflections off edges;

Corner paths——paths involving reflections off corners;

IV. 4 KINDS OF CLASSICAL PATHS

$$U^{\varepsilon_1\varepsilon_2\varepsilon_3}_{\mathbf{r},\mathbf{r}'} = -\frac{1}{2\pi^2}\sum_{l,m,n=-\infty}^{\infty}\frac{(-1)^n}{t^2+(2la+x+\varepsilon_1 x')^2+(2mb+y+\varepsilon_2 y')^2+(2nc+z+\varepsilon_3 z')^2} \tag{8}$$

then cylinder kernel for boundary conditions $(NN)(NN)(DN)$:

$$\overline{T}^{NNM}(t,\mathbf{r},\mathbf{r}') = U^{---}_{\mathbf{r},\mathbf{r}'} + U^{+--}_{\mathbf{r},\mathbf{r}'} + U^{-+-}_{\mathbf{r},\mathbf{r}'} + U^{--+}_{\mathbf{r},\mathbf{r}'} + U^{++-}_{\mathbf{r},\mathbf{r}'} + U^{-++}_{\mathbf{r},\mathbf{r}'} + U^{+-+}_{\mathbf{r},\mathbf{r}'} + U^{+++}_{\mathbf{r},\mathbf{r}'} \tag{9}$$

Periodic paths—— $(P \sim U^{---}_{\mathbf{r},\mathbf{r}'})$;*Corner paths*——$(C \sim U^{+++}_{\mathbf{r},\mathbf{r}'})$

Side paths——$(S_x \sim U^{+--}_{\mathbf{r},\mathbf{r}'})(S_y \sim U^{-+-}_{\mathbf{r},\mathbf{r}'})(S_z \sim U^{--+}_{\mathbf{r},\mathbf{r}'})$

Edge paths——$(E_{xy} \sim U^{++-}_{\mathbf{r},\mathbf{r}'})(E_{yz} \sim U^{-++}_{\mathbf{r},\mathbf{r}'})$ $(E_{xz} \sim U^{+-+}_{\mathbf{r},\mathbf{r}'})$

For $(NN)(DD)(DN)$:

$$\overline{T}^{NDM}(t,\mathbf{r},\mathbf{r}') = U^{---}_{\mathbf{r},\mathbf{r}'} + U^{+--}_{\mathbf{r},\mathbf{r}'} - U^{-+-}_{\mathbf{r},\mathbf{r}'} + U^{--+}_{\mathbf{r},\mathbf{r}'} - U^{++-}_{\mathbf{r},\mathbf{r}'} - U^{-++}_{\mathbf{r},\mathbf{r}'} + U^{+-+}_{\mathbf{r},\mathbf{r}'} - U^{+++}_{\mathbf{r},\mathbf{r}'} \tag{10}$$

For $(DD)(NN)(DN)$:

$$\overline{T}^{DNM}(t,\mathbf{r},\mathbf{r}') = U^{---}_{\mathbf{r},\mathbf{r}'} - U^{+--}_{\mathbf{r},\mathbf{r}'} + U^{-+-}_{\mathbf{r},\mathbf{r}'} + U^{--+}_{\mathbf{r},\mathbf{r}'} - U^{++-}_{\mathbf{r},\mathbf{r}'} + U^{-++}_{\mathbf{r},\mathbf{r}'} - U^{+-+}_{\mathbf{r},\mathbf{r}'} - U^{+++}_{\mathbf{r},\mathbf{r}'} \tag{11}$$

For $(DD)(DD)(DN)$:

$$\overline{T}^{DDM}(t,\mathbf{r},\mathbf{r}') = U^{---}_{\mathbf{r},\mathbf{r}'} - U^{+--}_{\mathbf{r},\mathbf{r}'} - U^{-+-}_{\mathbf{r},\mathbf{r}'} + U^{--+}_{\mathbf{r},\mathbf{r}'} + U^{++-}_{\mathbf{r},\mathbf{r}'} - U^{-++}_{\mathbf{r},\mathbf{r}'} - U^{+-+}_{\mathbf{r},\mathbf{r}'} + U^{+++}_{\mathbf{r},\mathbf{r}'} \tag{12}$$

V. ENERGY DENSITY AND TOTAL ENERGY

$$\varepsilon = -\frac{1}{2}\lim_{t\to 0}\frac{\partial^2}{\partial^2 t}\overline{T}(t,\mathbf{r},\mathbf{r}) \tag{13}$$

$$E = \int \varepsilon dV \tag{14}$$

Periodic paths

$$\begin{aligned} E_M^{---} &= -\frac{1}{2\pi^2}\sum_{l,m,n=-\infty}^{\infty}{}' \frac{(-1)^n abc}{[(2la)^2+(2mb)^2+(2nc)^2]^2} \\ &= -\frac{abc}{32\pi^2}[2Z_3(a,b,2c;4) - Z_3(a,b,c;4)] \end{aligned} \tag{15}$$

Side paths

$$\begin{aligned} E^{+--} &= -\frac{1}{2\pi^2}bc\sum_{m,n=-\infty}^{\infty}{}' \frac{(-1)^n}{[(2mb)^2+(2nc)^2]^{\frac{3}{2}}} \\ &= -\frac{bc}{64\pi}[2Z_2(b,2c;3) - Z_2(b,c;3)] \end{aligned} \tag{16}$$

$$\begin{aligned} E^{-+-} &= -\frac{1}{2\pi^2}ac\sum_{l,n=-\infty}^{\infty}{}' \frac{(-1)^n}{[(2la)^2+(2nc)^2]^{\frac{3}{2}}} \\ &= -\frac{ac}{64\pi}[2Z_2(a,2c;3) - Z_2(a,c;3)] \end{aligned} \tag{17}$$

$$E_M^{--+} = 0 \tag{18}$$

where $Z_d(a_1,..,a_d;s)$ is the Epstein zeta function.

VI. ENERGY DENSITY AND TOTAL ENERGY

Edge paths

$$E^{++-} = -\frac{c}{32\pi}\sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n}{n^2} = \frac{\pi}{192c} \tag{19}$$

$$E^{+-+} = 0 \tag{20}$$

$$E^{-++} = 0 \tag{21}$$

Corner paths

$$E^{+++} = 0 \tag{22}$$

$$E^{NNM} = E^{---} + E^{+--} + E^{-+-} + E^{--+} + E^{++-} + E^{-++} + E^{+-+} + E^{+++} \tag{23}$$

$$E^{NDM} = E^{---} + E^{+--} - E^{-+-} + E^{--+} - E^{++-} - E^{-++} + E^{+-+} - E^{+++} \tag{24}$$

$$E^{DNM} = E^{---} - E^{+--} + E^{-+-} + E^{--+} - E^{++-} + E^{-++} - E^{+-+} - E^{+++} \tag{25}$$

$$E^{DDM} = E^{---} - E^{+--} - E^{-+-} + E^{--+} + E^{++-} - E^{-++} - E^{+-+} + E^{+++} \tag{26}$$

Till now, we can deal with the rectangular cavity with any combination of D and N. Easily to be extended for pistons.

VII. AN ALTERNATIVE EXPRESSION FOR CYLINDER KERNEL

$$\overline{T}^{NNM} = -\sum_{l,m,n=-\infty}^{\infty} \frac{|D_{lmn}|^2}{\omega_{lmn}} cosk_1xcosk_2ysink_3zcosk_1x' cosk_2y' sink_3z' e^{-\omega_{lmn}t} \quad (27)$$

$$\overline{T}^{NDM} = -\sum_{l,m,n=-\infty}^{\infty} \frac{|D_{lmn}|^2}{\omega_{lmn}} cosk_1xsink_2ysink_3zcosk_1x' sink_2y' sink_3z' e^{-\omega_{lmn}t} \quad (28)$$

$$\overline{T}^{DNM} = -\sum_{l,m,n=-\infty}^{\infty} \frac{|D_{lmn}|^2}{\omega_{lmn}} sink_1xcosk_2ysink_3zsink_1x' cosk_2y' sink_3z' e^{-\omega_{lmn}t} \quad (29)$$

$$\overline{T}^{DDM} = -\sum_{l,m,n=-\infty}^{\infty} \frac{|D_{lmn}|^2}{\omega_{lmn}} sink_1xsink_2ycosk_3zsink_1x' sink_2y' cosk_3z' e^{-\omega_{lmn}t} \quad (30)$$

where $\omega_{lmn}^2 = k_1^2 + k_2^2 + k_3^2$ and $k_1 = \frac{l\pi}{a}, k_2 = \frac{m\pi}{b}, k_3 = \frac{(n+\frac{1}{2})\pi}{c}$. It follows from the normalization condition that:

$$|D_{lmn}|^2 = \frac{\epsilon_{0l}\epsilon_{0m}\epsilon_{0n}}{abc} \quad (31)$$

$\epsilon_{0i} = 1$ for $i = 0$ and $\epsilon_{0i} = 2$ otherwise.

If we stick to this way with the help of Poisson Sum Formula, we can reproduce the previous results $(9 - 12)$.

VIII. 3D RECTANGULAR PISTONS WITH MIXED BOUNDARY CONDITIONS——EM FIELD

Hertz Potential:

$$(\Phi, \overrightarrow{A}) = (-\nabla \cdot \overrightarrow{\Pi}_e, \partial_t \overrightarrow{\Pi}_e + \nabla \times \overrightarrow{\Pi}_m) \tag{32}$$

If we choose the Hertz potentials as $\overrightarrow{\Pi}_e = \varphi \overrightarrow{e}_3$ and $\overrightarrow{\Pi}_m = \psi \overrightarrow{e}_3$,then

$$(\Phi, \overrightarrow{A}) = (-\partial_3 \varphi, \partial_2 \psi, -\partial_1 \psi, +\partial_0 \varphi) \tag{33}$$

By Maxwell Equation:

$$\begin{aligned} \overrightarrow{E} &= (\partial_1 \partial_3 \phi - \partial_0 \partial_2 \psi, \partial_2 \partial_3 \phi + \partial_1 \partial_0 \psi, \partial_3^2 \phi - \partial_0^2 \phi) \\ \overrightarrow{B} &= (\partial_0 \partial_2 \phi + \partial_1 \partial_3 \psi, -\partial_0 \partial_1 \phi + \partial_2 \partial_3 \psi, \partial_3^2 \psi - \partial_0^2 \psi) \end{aligned} \tag{34}$$

Conducting Boundary Condition—$E_t = 0, B_n = 0$

Permeable Boundry Condition—$E_n = 0, B_t = 0$

We investigate the cavity with mixed boundary conditions: permeable boundary at $z = c$ and conducting boundary at other walls. The appropriate Hertz potentials are:

$$\begin{aligned} \phi_{lmn}(x, y, z) &= \sum_{l,m,n=-\infty}^{\infty} D_{lmn} \frac{sink_1 x sink_2 y cos k_3 z e^{-i\omega_{lmn} t}}{|k_{lmn}^{\perp}| \sqrt{2\omega_{lmn}}} \\ \psi_{lmn}(x, y, z) &= \sum_{l,m,n=-\infty}^{\infty} D_{lmn} \frac{cos k_1 x cos k_2 y sink_3 z e^{-i\omega_{lmn} t}}{|k_{lmn}^{\perp}| \sqrt{2\omega_{lmn}}} \end{aligned} \tag{35}$$

IX. ENERGY DENSITY AND TOTAL ENERGY

The energy density $\sim \frac{1}{2}(E^2 + B^2)$

$i\omega_{lmn}(t - t') \sim -\omega_{lmn}\tau$

$(k_1^2, k_2^2, k_3^2) \sim (-\partial_1^2, -\partial_2^2, -\partial_3^2,)$

$$E_3^2(t, \mathbf{x}, \mathbf{x}') = \sum_{l,m,n=-\infty}^{\infty} \frac{|D_{lmn}|^2(k_1^2 + k_2^2)}{2\omega_{lmn}} sink_1 x sink_2 y cosk_3 z sink_1 x' sink_2 y' cosk_3 z' e^{i\omega_{lmn}(t-t')} \tag{36}$$

$$\begin{aligned} E_3^2(\tau, \mathbf{x}, \mathbf{x}') &= \sum_{l,m,n=-\infty}^{\infty} \frac{|D_{lmn}|^2(k_1^2 + k_2^2)}{2\omega_{lmn}} sink_1 x sink_2 y cosk_3 z sink_1 x' sink_2 y' cosk_3 z' e^{-\omega_{lmn}\tau} \\ &= \frac{1}{2}(\partial_1^2 + \partial_2^2)\overline{T}^{DDM} \end{aligned} \tag{37}$$

$$\begin{aligned} \varepsilon^{5C1P}(\tau, \mathbf{r}, \mathbf{r}) =& \frac{1}{2}(E_3^2 + B_3^2 + E_2^2 + B_2^2 + E_1^2 + B_1^2) \\ =& \frac{1}{4}[(\partial_1^2 + \partial_2^2)\overline{T}^{DDM} + (\partial_2^2 + \partial_3^2)\overline{T}^{NDM} + (\partial_1^2 + \partial_3^2)\overline{T}^{DNM} \\ & + (\partial_1^2 + \partial_2^2)\overline{T}^{NNM} + (\partial_2^2 + \partial_3^2)\overline{T}^{DNM} + (\partial_1^2 + \partial_3^2)\overline{T}^{NDM}] \end{aligned} \tag{38}$$

$$\begin{aligned} \varepsilon^{5C1P}(\tau, \mathbf{r}, \mathbf{r}) = -\frac{1}{4}[&\partial_\tau^2(\overline{T}^{DDM} + \overline{T}^{NDM} + \overline{T}^{DNM} + \overline{T}^{NNM} + \overline{T}^{DNM} + \overline{T}^{NDM}) \\ & + \partial_1^2(\overline{T}^{NDM} + \overline{T}^{DNM}) + \partial_2^2(\overline{T}^{DNM} + \overline{T}^{NDM}) + \partial_3^2(\overline{T}^{DDM} + \overline{T}^{NNM})] \end{aligned} \tag{39}$$

X. ENERGY DENSITY AND TOTAL ENERGY

Substitute (9-12) into (39)

$$\varepsilon^{5C1P}(\tau, \mathbf{r}, \mathbf{r}) = -\partial_\tau^2 U^{---} - \partial_\tau^2 U^{--+} - \partial_3^2 U^{++-} - \partial_3^2 U^{+++} \tag{40}$$

Notice that:

$$\begin{aligned} \partial_3^2 U^{++-} &= \partial_\tau^2 U^{++-} - \frac{4}{\pi^2} \sum_{l,m,n=-\infty}^{\infty} (-1)^n \frac{(2nc)^2 - t^2}{(d_{lmn}^{E_{xy}})^6} \\ \partial_3^2 U^{+++} &= \partial_\tau^2 U^{+++} - \frac{4}{\pi^2} \sum_{l,m,n=-\infty}^{\infty} (-1)^n \frac{(2nc)^2 - t^2}{(d_{lmn}^{C})^6} \end{aligned} \tag{41}$$

Then:

$$\begin{aligned} \varepsilon^{5C1P}(\tau, \mathbf{r}, \mathbf{r}) = &- \partial_\tau^2 [U^{---} + U^{--+} + U^{++-} + U^{+++}] \\ &+ \frac{4}{\pi^2} \sum_{l,m,n=-\infty}^{\infty} (-1)^n \frac{(2nc)^2}{(d_{lmn}^{E_{xy}})^6} + \frac{4}{\pi^2} \sum_{l,m,n=-\infty}^{\infty} (-1)^n \frac{(2nc)^2}{(d_{lmn}^{C})^6} \end{aligned} \tag{42}$$

$$\begin{aligned} E^{5C1P} =& 2[E_M^{---} + E_M^{--+} + E_M^{++-} + E_M^{+++}] - \frac{\pi}{48c} + \frac{\pi}{72c} \\ =& 2[E_M^{---} + 0 + E_M^{++-} + 0] - \frac{\pi}{144c} \\ =& - \frac{abc}{16\pi^2} \sum_{l,m,n=-\infty}^{\infty'} \frac{(-1)^n}{[(la)^2 + (mb)^2 + (nc)^2]^2} + \frac{\pi}{96c} - \frac{\pi}{144c} \\ =& - \frac{abc}{16\pi^2} [2Z_3(a, b, 2c; 4) - Z_3(a, b, c; 4)] + \frac{\pi}{288c} \end{aligned} \tag{43}$$

POSITIVE!

XI. ENERGY DENSITY AND TOTAL ENERGY

To compare with the rectangular cavity with conducting boundary at every wall, which has energy

$$
\begin{aligned}
E^{Con} =& 2[E^{---} + E^{-++} + E^{+-+} + E^{++-}] + \frac{\pi}{24a} + \frac{\pi}{24b} + \frac{\pi}{24c} \\
=& E^{DDD} + E^{NNN} + \frac{\pi}{24a} + \frac{\pi}{24b} + \frac{\pi}{24c} \\
=& -\frac{abc}{16\pi^2} Z_3(a,b,c;4) + \frac{\pi}{48}(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})
\end{aligned}
\tag{44}
$$

Next to form a piston. E^{Con} is $x-y-z$ symmetrical, the force on the partition could be either attractive or repulsive to the nearest wall, depending on the ration $a:b:c$.

However, for E^{5C1P}, $x-y-z$ symmetry is destroyed but $x-y$ symmetry stays.

$$
E^{5C1P} = -\frac{abc}{16\pi^2}[2Z_3(a,b,2c;4) - Z_3(a,b,c;4)] + \frac{\pi}{288c} \tag{45}
$$

We have two ways to form a piston:

1. with the permeable wall (z=c) movable;

$$
F_c^{5C1P} = -\partial_c[E^{5C1P}(a,b,c) + E^{5C1P}(a,b,L-c)]|_{L\longrightarrow\infty} \tag{46}
$$

2. with the conducting wall movable.

$$
F_a^{5C1P} = -\partial_a[E^{5C1P}(a,b,c) + E^{5C1P}(L-a,b,c)]|_{L\longrightarrow\infty} \tag{47}
$$

For both cases, the force on the partition is always repulsive.