Two-dimensional Euclidean Helliwell–Konkowski Calculation

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I want to show how the approach of Helliwell and Konkowski [8] (derived from Deutsch and Candelas [6]) is related to the two formulas derived in `ckpc` and `tft3d`. Actually, I prefer to work entirely in the elliptic domain, with a cylinder kernel instead of a Green function for the wave equation. Equation numbers in brackets are pointers to the closest equivalents in [8].

Consider the operator [(4)]

$$H \equiv -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{1}$$

Note that t is a cylinder coordinate, not physical time. Now e^{-sH} is a heat operator with respect to an additional "time" coordinate s. Thus [(8)]

$$G(x, x') = \int_0^\infty ds\, e^{-sH}\delta(x, x') \tag{2}$$

is the integral kernel of the operator

$$\int_0^\infty e^{-sH}\, ds = \left.\frac{e^{-sH}}{-H}\right|_0^\infty = H^{-1}; \tag{3}$$

this calculation is meaningful when H (is essentially self-adjoint and) has no negative spectrum and no zero eigenvalue.

The (generalized) eigenfunctions of H are [(6)]

$$u(x, \omega, p, n) = \sqrt{\frac{p}{\theta_1}} J_{|\lambda|}(pr) e^{i\omega t} e^{i\lambda\theta} \tag{4}$$

where

$$\lambda = \lambda_n \equiv \frac{2\pi n}{\theta_1} \tag{5}$$

and the eigenvalue is $-(\omega^2+p^2)$. (Following [8] I have included the normalization factor for the discrete θ eigenfunction but left the 2π for the ω integration in the next step.) Therefore [(7)]

$$\delta(x,x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^\infty dp \sum_{n=-\infty}^{\infty} u(x)u^*(x'). \tag{6}$$

Combining (6), (2), and (4), we get [(9) or (10)]

$$G(x,x') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^\infty p\,dp \int_0^\infty ds\, \frac{1}{\theta_1} \sum_{n=-\infty}^{\infty} e^{-s(\omega^2+p^2)} e^{i\omega(t-t')} J_{|\lambda|}(pr) J_{|\lambda|}(pr') e^{i\lambda(\theta-\theta')}. \tag{7}$$

Now let's consider strategies for evaluating, or at least simplifying, this expression.

Approach 1: Do the s integral first. In effect, this undoes steps (2) and (3) to construct the integral kernel of H^{-1} directly. We get

$$G(x,x') = \frac{1}{2\pi\theta_1} \int_{-\infty}^{\infty} d\omega \int_0^\infty p\,dp \sum_{n=-\infty}^{\infty} \frac{1}{\omega^2+p^2} e^{i\omega(t-t')} J_{|\lambda|}(pr) J_{|\lambda|}(pr') e^{i\lambda(\theta-\theta')}. \tag{8}$$

More directly: The Green function should satisfy $H_x G(x,x') = \delta(x,x')$, which is

$$\frac{\partial^2 G}{\partial t^2} + \frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} = -\frac{1}{r}\delta(r-r')\delta(\theta-\theta')\delta(t-t'). \tag{9}$$

This is equivalent to (1) of `tft3d`, since $\overline{T} = -2G|_{t'=0}$. Let us solve (9) by a *complete* Fourier analysis:

$$\hat{G}(\omega,p,n,x') \equiv \frac{1}{2\pi} \int_{\infty}^{\infty} d\omega\, e^{-i\omega t} \int_0^\infty r\,dr\, J_{|\lambda|}(pr) \frac{1}{\theta_1} \int_0^{\theta_1} d\theta\, e^{-\lambda\theta} G(x,x'), \tag{10}$$

whence

$$-\omega^2 \hat{G} - p^2 \hat{G} = -\frac{1}{2\pi\theta_1} e^{-i\omega t'} J_{|\lambda|}(pr') e^{-i\lambda\theta'}. \tag{11}$$

Invert all the transforms:

$$G(t,r,\theta,x') = \int_{-\infty}^{\infty} d\omega \int_0^\infty p\,dp \sum_{n=-\infty}^{\infty} e^{i\omega t} J_{|\lambda|}(pr) e^{i\lambda\theta} \hat{G}(\omega,p,n,x'). \tag{12}$$

From (12) and (11), (8) results immediately.

There are two ways to press onward:

Approach 1(a): Do the ω integral in (8) next. We can assume that $t - t' > 0$, since for the standard cylinder kernel we need to take $t' = 0$, $t = 0$. Then we can close the contour in the upper half plane, and since

$$\omega^2 + p^2 = (\omega + ip)(\omega - ip),$$

there is a pole at $\omega = ip$ and the other factor becomes $2ip$. After cancelling $2\pi i$, etc., we have

$$G = \frac{1}{2\theta_1} \int_0^\infty dp\, e^{-p(t-t')} J_{|\lambda|}(pr) J_{|\lambda|}(pr') e^{i\lambda(\theta-\theta')}, \tag{13}$$

which is essentially the result of solving the cylinder-kernel problem by separation of variables: see (25) of `ckpc` and the remark following it.

Approach 1(b): Do the p integral next. The relevant integral is Gradshteyn–Ryzhik 6.541.1:

$$\int_0^\infty pJ_\nu(rp) j_\nu(r'p) \frac{dp}{p^2+\omega^2} = I_\nu(|\omega| r_<) K_\nu(|\omega| r_>). \tag{14}$$

It converts (8) to

$$G(x,x') = \frac{1}{2\pi\theta_1} \int_{-\infty}^\infty d\omega \sum_{n=-\infty}^{\infty} e^{i\omega(t-t')} I_{|\lambda|}(|\omega| r_<) K_{|\lambda|}(|\omega| r_>) e^{i\lambda(\theta-\theta')}, \tag{15}$$

which is equivalent to formula (14) of `tft3d`.

Approach 2: In (7) do the ω integral first. This is the route taken in [8] and [6]; those authors do a polar integral in (ω, k) space, but we have no k. As anticipated, they get a more closed final form in 3D than we can get in 2D. Using

$$\frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, e^{-s\omega^2} e^{i\omega(t-t')} = \frac{1}{\sqrt{4\pi s}} e^{-(t-t')/4s}, \tag{16}$$

I get

$$G = \int_0^\infty p\, dp \int_0^\infty \frac{ds}{\sqrt{s}} \frac{1}{2\theta_1\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t-t')/4s} e^{-sp^2} J_{|\lambda|}(pr) J_{|\lambda|}(pr') e^{i\lambda(\theta-\theta')}. \tag{17}$$

I suspect that either the p or the s integral or both can be evaluated by known formulas, but I shall stop here for today.