# Constructing cylinder kernels by Fourier transform in $t$

I will work out the example of the free field in two-dimensional space-time. Consider the Green function equation

$$\frac{\partial^2 \overline{T}}{\partial t^2} + \frac{\partial^2 \overline{T}}{\partial x^2} = 2\delta(t)\delta(x - x'). \tag{1}$$

(Why this equation is satisfied by our "second" cylinder kernel will be explained at the end. For now just note that the operator on the left is the elliptic one, the same as in our usual cylinder equation.)

Define the Fourier transform in $t$ by

$$\tilde{T}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} T(t)\, dt.$$

Note that $\omega$ may be negative. Then (1) becomes

$$-\omega^2 \tilde{T} + \frac{\partial^2 \tilde{T}}{\partial x^2} = 2\delta(x - x'). \tag{2}$$

The solution has the form

$$\tilde{T}(\omega, x, x') = \begin{cases} C_\omega e^{|\omega| x} & \text{if } x < x', \\ D_\omega e^{-|\omega| x} & \text{if } x > x'. \end{cases} \tag{3}$$

Continuity implies

$$D_\omega = C_\omega e^{2|\omega| x'}.$$

The jump condition is

$$2 = D_\omega(-|\omega|)e^{-|\omega| x'} - C_\omega(|\omega|)e^{|\omega| x'} = -2|\omega| C_\omega e^{|\omega| x'}.$$

Therefore,

$$C_\omega = -\frac{1}{|\omega|} e^{-|\omega| x'}, \qquad D_\omega = -\frac{1}{|\omega|} e^{|\omega| x'},$$

so from (3)

$$\tilde{T}(\omega, x, x') = -\frac{1}{|\omega|} e^{-|\omega| |x - x'|}. \tag{4}$$

If we try to invert the transform, we get

$$\overline{T}(t, x, x') = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|\omega|} e^{-|\omega| |x - x'|} e^{i\omega t}\, d\omega. \tag{5}$$

This integral is ill-defined at $\omega = 0$. First look instead at

$$
\begin{aligned}
T(t, x, x') &= \frac{\partial \overline{T}}{\partial t} \\
&= -\frac{i}{2\pi} \int_{-\infty}^{\infty} \operatorname{sgn} \omega \, e^{-|\omega||x-x'|} e^{i\omega t} \, d\omega \\
&= \frac{1}{\pi} \int_0^{\infty} \sin(\omega t) e^{-\omega|x-x'|} \, d\omega \\
&= \frac{t/\pi}{t^2 + (x-x')^2} \; .
\end{aligned}
\tag{6}
$$

This is the expected answer.

Furthermore, the expected $\overline{T}$ is

$$
\overline{T}(t, x, x') = \frac{1}{2\pi} \ln[t^2 + (x-x')^2],
\tag{7}
$$

up to an additive constant (corresponding to the scale ambiguity in the argument of the logarithm), which is a feature peculiar to two-dimensional massless theories. Let us try to verify the transform in the other direction. The only relevant integral in the Gradshteyn–Ryzhik tables is (4.382.3):

$$
\int_0^{\infty} \ln \frac{a^2 + x^2}{b^2 + x^2} \cos(cx) \, dx = \frac{\pi}{c} \left( e^{-bc} - e^{-ac} \right) \quad (a, b, c > 0).
\tag{8}
$$

This can be regarded as the transform of the *difference* of the Green function at two different values of $|x - x'|$. (The convergence of the integral at $\infty$ would be poor for each term by itself.) Define

$$
\Delta \tilde{T} = \int_{-\infty}^{\infty} e^{-i\omega t} dt \, \frac{1}{2\pi} \left[ \ln[t^2 + (x-x')^2] - \ln[t^2 + R^2] \right] \quad (R > 0).
\tag{9}
$$

For $\omega > 0$, (8) gives

$$
\Delta \tilde{T} = \frac{1}{\omega} \left( e^{-\omega R} - e^{-\omega|x-x'|} \right),
\tag{10}
$$

and (9) is obviously even in $\omega$. Thus (10) is consistent with (4).

Finally, the justification of (1) (in a more general context) is in my handwritten notes of 18 May 2000, "Preliminaries to Numerical Analysis of Cylinder Kernels", pp. 6–8. Actually, there I worked with $T$, but the argument for $\overline{T}$ is similar. In brief, the factor 2 arises from an application of the method of images on the surface $t = 0$.

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