[based on tft.pdf by S. A. Fulling]
Consider the Green function:

$$\frac{\partial^2 \bar{T}}{\partial t^2} + \frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{T}}{\partial \theta^2} = \frac{2}{r}\delta(t)\delta(r-r^{'})\delta(\theta-\theta^{'}) \tag{1}$$

Define a Fourier transformation in θ:

$$\bar{T}(t,r,\theta) = \frac{1}{\theta_1}\sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})}T_n(t,r) \tag{2}$$

$$T_n(t,r) = \int_0^{\theta_1} e^{-in\theta(\frac{2\pi}{\theta_1})}\bar{T}(t,r,\theta) \tag{3}$$

Let $\lambda = \frac{2n\pi}{\theta_1}$ and take $\theta^{'} = 0$ WLOG, hence:

$$\frac{\partial^2 T_n}{\partial t^2} + \frac{\partial^2 T_n}{\partial r^2} + \frac{1}{r}\frac{\partial T_n}{\partial r} - \frac{\lambda^2}{r^2}T_n = \frac{2}{r}\delta(t)\delta(r-r^{'}) \tag{4}$$

Now do a Fourier transform in t:

$$\tilde{T}(\omega,r) = \int_{-\infty}^{\infty} e^{-i\omega t}T_n(t,r)dt \tag{5}$$

So that (4) becomes:

$$-\omega^2\tilde{T} + \frac{\partial^2 \tilde{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{T}}{\partial r} - \frac{\lambda^2}{r^2}\tilde{T} = \frac{2}{r}\delta(r-r^{'}) \tag{6}$$

The solution has the form:

$$\tilde{T} = \begin{cases} C_\omega I_{|\lambda|}(|\omega|r) & \text{for} \quad \mathrm{r} < \mathrm{r}^{'} \\ D_\omega K_{|\lambda|}(|\omega|r) & \text{for} \quad \mathrm{r} > \mathrm{r}^{'} \end{cases} \tag{7}$$

Continuity condition implies

$$D_\omega = C_\omega \frac{I_{|\lambda|}(|\omega|r')}{K_{|\lambda|}(|\omega|r')} \tag{8}$$

Jump condition implies

$$\frac{2}{r'} = D_\omega|\omega|\frac{\partial K_{|\lambda|}}{\partial r}|_{|\omega|r'} - C_\omega|\omega|\frac{\partial I_{|\lambda|}}{\partial r}|_{|\omega|r'} = C_\omega|\omega|(\frac{I_{|\lambda|}}{K_{|\lambda|}}\frac{\partial K_{|\lambda|}}{\partial r} - \frac{\partial I_{|\lambda|}}{\partial r})|_{|\omega|r'} \tag{9}$$

Solving for C_ω and D_ω, using the Wronskian determinant, $I_m(x)K_m'(x) - I_m'(x)K_m(x) = -\frac{1}{x}$, we have

$$C_\omega = -2K_{|\lambda|}(|\omega|r') \tag{10}$$

$$D_\omega = -2I_{|\lambda|}(|\omega|r') \tag{11}$$

So

$$\tilde{T}(|\omega|, r) = -2I_{|\lambda|}(|\omega|r_<)K_{|\lambda|}(|\omega|r_>) \tag{12}$$

Hence we can invert the Fourier transform:

$$T_n(t, r) = -\frac{1}{\pi}\int_{-\infty}^{\infty} I_{|\lambda|}(|\omega|r_<)K_{|\lambda|}(|\omega|r_>)e^{i|\omega|t}d\omega \tag{13}$$

$$\bar{T}(t, r, \theta) = -\frac{1}{\pi\theta_1}\sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})}\int_{-\infty}^{\infty} I_{|\lambda|}(|\omega|r_<)K_{|\lambda|}(|\omega|r_>)e^{i|\omega|t}d\omega \tag{14}$$

Following Smith, the ω integral gives a Legendre function of the second kind which can be transformed in another integral

$$2\int_{0}^{\infty} I_{|\lambda|}(|\omega|r_<)K_{|\lambda|}(|\omega|r_>)e^{i|\omega|t}d\omega = \frac{1}{\sqrt{rr'}}Q_{|\lambda|-\frac{1}{2}}(\cosh u_0) \tag{15}$$

$$= \frac{1}{\sqrt{2rr'}}\int_{u_0}^{\infty} du\frac{e^{-|\lambda|u}}{\sqrt{\cosh u - \cosh u_0}} \tag{16}$$

where $\cosh u_0 = \frac{r^2+{r'}^2+t^2}{2rr'}$. Our final answer is:

$$\bar{T}(t, r, \theta) = -\frac{1}{\pi\theta_1\sqrt{2rr'}}\int_{u_0}^{\infty} du\frac{1}{\sqrt{\cosh u - \cosh u_0}} \times \sum_{n=-\infty}^{\infty} e^{-|\lambda|u}e^{i\lambda\theta} \tag{17}$$

$$= -\frac{1}{\pi\theta_1\sqrt{2rr'}}\int_{u_0}^{\infty} du\frac{1}{\sqrt{\cosh u - \cosh u_0}}\frac{\sinh(\frac{2\pi}{\theta_1}u)}{\cosh(\frac{2\pi}{\theta_1}u) - \cos(\frac{2\pi\theta}{\theta_1})} \tag{18}$$