[based on tft.pdf by S. A. Fulling and Gravitational Effect... by A. G. Smith]
Consider the Green function:

$$\frac{\partial^2 \bar{T}}{\partial t^2} + \frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{T}}{\partial \theta^2} + \frac{\partial^2 \bar{T}}{\partial z^2} = \frac{2}{r}\delta(t)\delta(r-r^{'})\delta(\theta-\theta^{'})\delta(z-z') \tag{1}$$

Define a Fourier transformation in θ:

$$\bar{T}(t,r,\theta,z) = \sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})} T_n(t,r,z) \tag{2}$$

$$T_n(t,r,z) = \frac{1}{\theta_1}\int_0^{\theta_1} e^{-in\theta(\frac{2\pi}{\theta_1})}\bar{T}(t,r,\theta,z) \tag{3}$$

Let $\lambda = \frac{2n\pi}{\theta_1}$, take $\theta^{'}=0$ and $z'=0$ WLOG, hence:

$$\frac{\partial^2 T_n}{\partial t^2} + \frac{\partial^2 T_n}{\partial r^2} + \frac{1}{r}\frac{\partial T_n}{\partial r} - \frac{\lambda^2}{r^2}T_n + \frac{\partial^2 T_n}{\partial z^2} = \frac{2}{\theta_1 r}\delta(t)\delta(r-r')\delta(z) \tag{4}$$

The delta function in z and t can be represented as

$$\delta(z-z') = \frac{2}{\pi}\int_0^\infty dk \cos kz \cos kz' \tag{5}$$

$$\delta(t-t') = \frac{2}{\pi}\int_0^\infty d\omega' \cos\omega' t\cos\omega' t' \tag{6}$$

So one can have

$$\tilde{T}_n(\omega',r,k) = \int_0^\infty dt'\cos(\omega' t')\int_0^\infty dz'\cos(kz')T_n(t',r,z') \tag{7}$$

$$T_n(t,r,z) = \frac{2}{\pi}\int_0^\infty d\omega'\cos(\omega' t)\frac{2}{\pi}\int_0^\infty dk\cos(kz)\tilde{T}_n(\omega',r,k) \tag{8}$$

So that (4) becomes:

$$-(\omega^{'2}+k^2)\tilde{T}_n + \frac{\partial^2 \tilde{T}_n}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{T}_n}{\partial r} - \frac{\lambda^2}{r^2}\tilde{T}_n = \frac{2}{\theta_1 r}\delta(r-r') \tag{9}$$

Let $\omega^2 = \omega^{'2}+k^2$. The solution has the form:

$$\tilde{T}_n = \begin{cases} C_\omega I_{|\lambda|}(\omega r) & \text{for} \quad \mathrm{r} < \mathrm{r}^{'} \\ D_\omega K_{|\lambda|}(\omega r) & \text{for} \quad \mathrm{r} > \mathrm{r}^{'} \end{cases} \tag{10}$$

Continuity condition implies

$$D_\omega = C_\omega \frac{I_{|\lambda|}(\omega r')}{K_{|\lambda|}(\omega r')} \tag{11}$$

Jump condition implies

$$-\frac{2}{\theta_1 r'} = D_\omega \omega\frac{\partial K_{|\lambda|}}{\partial r}|_{\omega r'} - C_\omega\omega\frac{\partial I_{|\lambda|}}{\partial r}|_{\omega r'} = C_\omega\omega(\frac{I_{|\lambda|}}{K_{|\lambda|}}\frac{\partial K_{|\lambda|}}{\partial r} - \frac{\partial I_{|\lambda|}}{\partial r})|_{\omega r'} \tag{12}$$

Solving for C_ω and D_ω, using the Wronskian determinant, $I_m(x)K'_m(x) - I'_m(x)K_m(x) = -\frac{1}{x}$, we have

$$C_\omega = -\frac{2}{\theta_1}K_{|\lambda|}(\omega r') \tag{13}$$

$$D_\omega = -\frac{2}{\theta_1}I_{|\lambda|}(\omega r') \tag{14}$$

So

$$\tilde{T}_n(\omega, r) = -\frac{2}{\theta_1}I_{|\lambda|}(\omega r_<)K_{|\lambda|}(\omega r_>) \tag{15}$$

Hence we get:

$$\bar{T}(t,r,\theta,z) = -\frac{2}{\pi^2\theta_1}\sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})} \times \int_0^\infty dk \int_0^\infty d\omega' \cos(\omega' t)\cos(kz) I_{|\lambda|}(\omega r_<)K_{|\lambda|}(\omega r_>) \tag{16}$$

Since ${\omega'}^2 = \omega^2 - k^2$, $d\omega' = \omega d\omega(\omega^2 - k^2)^{-\frac{1}{2}}$. Then

$$\begin{aligned}\bar{T}(t,r,\theta,z) &= -\frac{2}{\pi^2\theta_1}\sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})}\int_0^\infty \omega d\omega I_{|\lambda|}(\omega r_<)K_{|\lambda|}(\omega r_>) \\ &\quad \times \int_0^\omega dk \frac{\cos((\omega^2-k^2)^{\frac{1}{2}}t)\cos(kz)}{(\omega^2-k^2)^{\frac{1}{2}}}\end{aligned} \tag{17}$$

From (3.876.7) of Gradshteyn–Ryzhik, the k integral gives $\frac{\pi}{2}J_0(\omega\zeta)$, where $\zeta \equiv \sqrt{z^2+t^2}$. Thus,

$$\bar{T}(t,r,\theta,z) = -\frac{1}{\pi\theta_1}\sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})}\int_0^\infty \omega d\omega I_{|\lambda|}(\omega r_<)K_{|\lambda|}(\omega r_>)J_0(\omega\zeta) \tag{18}$$

$$= -\frac{1}{\pi\theta_1}\sum_{n=-\infty}^{\infty} e^{in\theta(\frac{2\pi}{\theta_1})} \times \frac{e^{-i\pi/2}}{\sqrt{2\pi}}\frac{Q^{\frac{1}{2}}_{|\lambda|-\frac{1}{2}}(\cosh u)}{rr'(\sinh u)^{\frac{1}{2}}} \tag{19}$$

from (6.578.11) of Gradshteyn–Ryzhik, where $\cosh u \equiv \frac{r^2+{r'}^2+z^2+t^2}{2rr'}$ and $Q^{\frac{1}{2}}_{|\lambda|-\frac{1}{2}}(\cosh u) = i\sqrt{\frac{\pi}{2\sinh u}}e^{-|\lambda|u}$ (from (8.754.4) of Gradshteyn–Ryzhik). So

$$\bar{T}(t,r,\theta,z) = -\frac{1}{2\pi\theta_1 rr'\sinh u}\sum_{n=-\infty}^{\infty} e^{i\lambda\theta-|\lambda|u} \tag{20}$$

$$= -\frac{1}{2\pi\theta_1 rr'\sinh u}\frac{\sinh\frac{2\pi u}{\theta_1}}{\cosh\frac{2\pi u}{\theta_1} - \cos\frac{2\pi\theta}{\theta_1}} \tag{21}$$

When $\theta_1 = 2\pi$,

$$\bar{T}(t,r,\theta,z) = -\frac{1}{4\pi^2 rr'(\cosh u - \cos\theta)} \tag{22}$$

$$= -\frac{1}{2\pi^2(r^2 + r'^2 - 2rr'\cos\theta + z^2 + t^2)} \tag{23}$$

which agrees with the end result in ckpc4d.