Hertz Potentials in Cylindrical Coordinates

Jeff Bouas

26 June 2009

Motivation

Motivation

1. To extend Hertz Potentials to general curvilinear coordinates.
2. To explore the usefulness of differential forms.

Differential Forms and the Hodge Star

Why use differential forms?

1. Maxwell's equations take a more elegant form.
2. Differential forms are intrinsically coordinate-independent.

Differential Forms and the Hodge Star

Definition

For a pseudo-Riemannian orientable metrizable manifold (M, g), the **Hodge Star** is the unique operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ such that for $\omega, \eta \in \Omega^k(M)$

$$\langle \omega, \eta \rangle = \int \omega \wedge *\eta$$

Definition

For 4-dimensional Minkowski space with positive signature, define the codifferential δ as

$$\delta = *d*$$

Maxwell's Equations

Let F be the 2-form representing the electromagnetic field in vacuum.

Theorem

Maxwell's equations then become

$$\begin{aligned} dF &= 0 \\ \delta F &= 0 \end{aligned}$$

Electromagnetic Potential

Lemma

For any form ω,

$$d^2\omega = 0$$

Since $dF = 0$, F is called **closed**, and for any simply connected manifold, every closed form is **exact**.
Thus there exists a 1-form $A = A_\mu dx^\mu$ such that $F = dA$.
Immediately from this it follows

$$dF = d^2 A = 0.$$

It also follows that for $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Hertz Potentials

Choosing the Lorentz gauge is equivalent to choosing $\delta A = 0$. Since $\delta^2 = \pm * d^2 *$, this means that $*A$ is closed and exact.

There exists a 2-form $\Pi = \frac{1}{2}\Pi_{\mu\nu} dx^\mu \wedge dx^\nu$ such that $\delta\Pi = A$.

Consequently, $F = dA = d\delta\Pi$.

Hertz Potentials (Cont.)

Definition

Define the operator $\Box$ such that

$$\Box\omega = d\delta\omega + \delta d\omega$$

The condition that $F = d\delta\Pi$ solves both Maxwell equations is satisfied when

$$\Box\Pi = d\delta\Pi + \delta d\Pi = 0.$$

Then $F = d\delta\Pi = -\delta d\Pi$.

Notation

Choose a particular coordinate system x^0, x^1, x^2, x^3.

Definition

Let $h^{\mu}{}_{\nu\rho\sigma}$, $h^{\mu\nu}{}_{\rho\sigma}$, and $h^{\mu\nu\rho}{}_{\sigma}$ such that

$$\begin{aligned} *dx^{\mu} &= h^{\mu}{}_{\nu\rho\sigma} dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} \\ *(dx^{\mu} \wedge dx^{\nu}) &= h^{\mu\nu}{}_{\rho\sigma} dx^{\rho} \wedge dx^{\sigma} \\ *(dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}) &= h^{\mu\nu\rho}{}_{\sigma} dx^{\sigma}. \end{aligned}$$

In Cartesian coordinates, these reduce to factors times $\epsilon_{\mu\nu\rho\sigma}$.

Potentials

In these arbitrary but fixed coordinates, the 4-potential and Hertz potential become

$$\Pi = \frac{1}{2}\Pi_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$$

$$A = A_{\mu} dx^{\mu} = \frac{1}{2}\partial_{\nu}(\Pi_{\rho\sigma} h^{\rho\sigma}{}_{\lambda\xi}) h^{\nu\lambda\xi}{}_{\mu} dx^{\mu}.$$

Field Equations

With the coordinates chosen and the previous definitions, the field becomes

$$F = \frac{1}{2}\partial_\mu(\partial_\lambda(\Pi_{\xi\delta}h^{\xi\delta}{}_{\rho\sigma})h^{\lambda\rho\sigma}{}_\nu)dx^\mu \wedge dx^\nu.$$

The condition $\Box\Pi = 0$ also becomes

$$\Box\Pi = \frac{1}{2}[\partial_\mu(h^{\lambda\rho\sigma}{}_\nu\partial_\lambda(h^{\xi\delta}{}_{\rho\sigma}\Pi_{\xi\delta}))+h^{\lambda\sigma}{}_{\mu\nu}\partial_\lambda(h^{\rho\xi\delta}{}_\sigma\partial_\rho\Pi_{\xi\delta})]dx^\mu\wedge dx^\nu = 0.$$

Unidirectional Hertz Potentials

Take $\Pi = \Pi_{01} dx^0 \wedge dx^1 + \Pi_{23} dx^2 \wedge dx^3$.

For $\Box\Pi = 0$, the $dx^0 \wedge dx^1$ and $dx^2 \wedge dx^3$ components yield the following equations.

$$\begin{aligned}
(\partial_0(h^{023}{}_1\partial_0) - \partial_1(h^{123}{}_0\partial_1))(h^{01}{}_{23}\Pi_{01}) + h^{23}{}_{01}(\partial_2(h^{201}{}_3\partial_2) - \partial_3(h^{301}{}_2\partial_3))\Pi_{01} &= 0 \\
h^{01}{}_{23}(\partial_0(h^{023}{}_1\partial_0) - \partial_1(h^{123}{}_0\partial_1))\Pi_{23} + (\partial_2(h^{201}{}_3\partial_2) - \partial_3(h^{301}{}_2\partial_3))(h^{23}{}_{01}\Pi_{23}) &= 0
\end{aligned}$$

To contrast, for a scalar field ϕ,

$$\Box\phi = [h^{0123}\partial_0(h^0{}_{123}\partial_0) + h^{1023}\partial_1(h^1{}_{023}\partial_1) + h^{2013}\partial_2(h^2{}_{013}\partial_2) + h^{3012}\partial_3(h^3{}_{012}\partial_3)]\phi$$

Cartesian Coordinates

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and let $\Pi_{01} = \phi$, $\Pi_{23} = \psi$.

Results

$$\Box \phi = 0$$

$$\Box \psi = 0$$

Cylindrical Coordinates

Let $x^0 = t$, $x^1 = z$, $x^2 = \rho$, $x^3 = \varphi$, and take $\Pi_{01} = \phi$, $\Pi_{23} = \psi \cdot \rho$.

$$\begin{aligned} \partial_t^2 \Pi_{01} - \frac{1}{\rho}\partial_\rho(\rho \partial_\rho \Pi_{01}) - \frac{1}{\rho^2}\partial_\varphi^2 \Pi_{01} - \partial_z^2 \Pi_{23} &= 0 \\ \partial_t^2 \Pi_{23} - \partial_\rho(\rho \partial_\rho \frac{\Pi_{23}}{\rho}) - \frac{1}{\rho^2}\partial_\varphi^2 \Pi_{23} - \partial_z^2 \Pi_{23} &= 0 \end{aligned}$$

Results

$$\Box \phi = 0$$

$$\Box \psi = 0$$

Cylindrical Coordinates (Again)

Now take $x^0 = t$, $x^1 = \rho$, $x^2 = \varphi$, $x^3 = z$, and $\Pi_{01} = \frac{\phi}{\rho}$, $\Pi_{23} = \psi$.

$$\begin{aligned}
\partial_t^2 \Pi_{01} - \partial_\rho\left(\frac{1}{\rho}\partial_\rho(\rho\Pi_{01})\right) - \frac{1}{\rho^2}\partial_\varphi^2 \Pi_{01} - \partial_z^2 \Pi_{01} &= 0 \\
\partial_t^2 \Pi_{23} - \rho\partial_\rho\left(\frac{1}{\rho}\partial_\rho \Pi_{23}\right) - \frac{1}{\rho^2}\partial_\varphi^2 \Pi_{23} - \partial_z^2 \Pi_{23} &= 0
\end{aligned}$$

Results

$$(\Box - \frac{2}{\rho}\partial_\rho)\phi = 0$$

$$(\Box - \frac{2}{\rho}\partial_\rho)\psi = 0$$

Spherical

Let $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ with $\Pi_{01} = \phi$, $\Pi_{23} = \psi \cdot r^2 \sin\theta$.

$$\begin{aligned} \partial_t^2 \Pi_{01} - \partial_r(\frac{1}{r^2}\partial_r(r^2\Pi_{01})) - \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Pi_{01}) - \frac{1}{r^2\sin\theta}\partial_\varphi^2\Pi_{01} &= 0 \\ \partial_t^2 \Pi_{23} - r^2\partial_r(\frac{1}{r^2}\partial_r\Pi_{23}) - \frac{1}{r^2}\partial_\theta(\sin\theta\partial_\theta(\frac{\Pi_{23}}{\sin\theta}) - \frac{1}{r^2\sin\theta}\partial_\varphi^2\Pi_{23} &= 0 \end{aligned}$$

Results

$$(\Box - \frac{2}{r^2})\phi = 0$$

$$(\Box - \frac{2}{r^2})\psi = 0$$