

# Integral Equations and Vacuum Energy

Quantum Vacuum Workshop, TAMU, June 2009

# Aims

- Calculate cylinder kernels directly (skipping eigenvalue density, heat kernel, etc.).
- Exploit theory of Volterra and Fredholm integral equations and possibly Padé approximants.
- Use multiple reflection expansion to *derive* and *improve upon*
  - optical approximation (for smooth boundaries)
  - geometrical diffraction theory (for nonsmooth ones).

# Kernels: Their Virtues and Vices

HEAT KERNEL  $e^{-tH}$

- rapid spectral convergence; smooth
- rapid decay & image convergence
- functoriality (product manifolds; mass dependence)
- exact WKB solution in free case
- initial value problem (Volterra)
- remote from quantum & wave physics
- Gutzwiller & Casimir info exponentially suppressed

# QUANTUM KERNEL $e^{-itH}$

- analytic continuation of heat kernel
- functoriality (product manifolds; mass dependence)
- exact WKB solution in free case
- initial value problem (Volterra)
  - remote from wave physics
  - oscillatory; distributional in general
  - poor decay & image convergence

## CYLINDER KERNEL $e^{-t\sqrt{H}}$

- Gives vacuum energy directly!
- rapid spectral convergence; smooth
  - mediocre decay & image convergence
  - no nice functorial properties
  - very poor WKB properties
  - not Volterra

## RESOLVENT KERNEL $(H - \lambda)^{-1}$

- Gives spectral information directly! “universal”
- one fewer dimension!
  - remote from physics unless  $\lambda > 0$
  - oscillatory ( $\lambda > 0$ )
  - poor decay & image convergence ( $\lambda > 0$ )
  - no nice functorial properties
  - mediocre WKB properties
  - not Volterra

## ZETA FUNCTION $H^{-s}$

- amenable to complex analysis
- hard to calculate directly by nonspectral means
- Gutzwiller & Casimir info suppressed  
(not in residues)

## WAVE KERNEL (D'ALEMBERT) $\cos(t\sqrt{H})$

- initial value problem (Volterra)
- finite range: very fast image convergence (for fixed  $t$ ).
- always distributional (not smooth)
- no nice functorial properties
- Casimir info hard to extract



# WAVE KERNEL (WIGHTMAN, FEYNMAN, SCHWINGER)

$$\frac{e^{-it\sqrt{H}}}{\sqrt{H}}$$

- analytic continuation of cylinder kernel
- Gives vacuum energy fairly directly.
- central to QFT (2=point function)
- oscillatory; distributional in general
- not Volterra

# Integral Equations

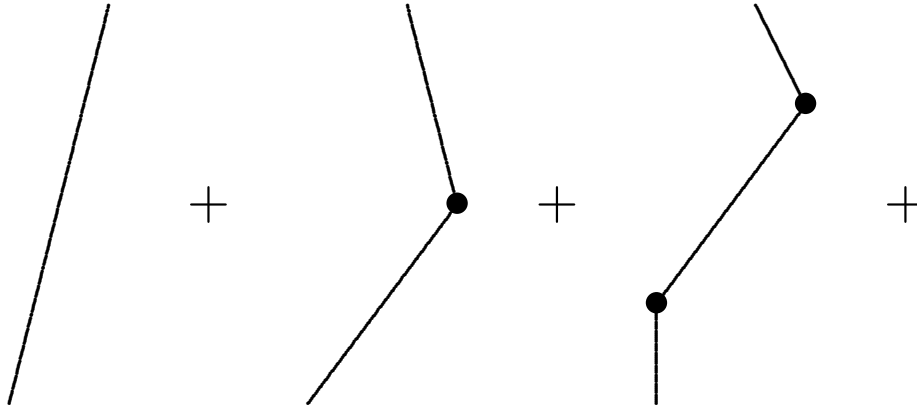
## VOLTERRA

Example:  $\int_0^t \lambda K(t, s)u(s) ds - u(t) = -f(t)$ .

$$u = (I - \lambda K)^{-1}(f) \sim \sum_{n=0}^{\infty} (\lambda K)^n(f).$$

$$\|K^n\| \leq \frac{1}{n!} \|K\|^n \Rightarrow \text{convergence for all } \lambda.$$

Intuitive interpretation: Path integral (perturbative).



## FREDHOLM

Example:  $\int_0^T \lambda K(t, s)u(s) ds - u(t) = -f(t)$ .

$$u = (I - \lambda K)^{-1}(f) \sim \sum_{n=0}^{\infty} (\lambda K)^n(f).$$

$$\|K^n\| \leq \|K\|^n \Rightarrow \text{convergence for } |\lambda| < \|K\|^{-1}$$

(and maybe on circle). But:

Generalized Cramer rule:

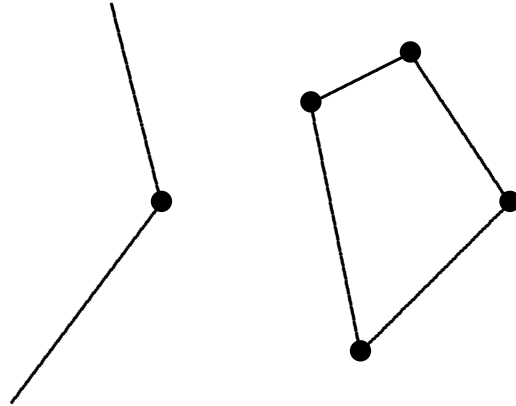
$$u(t) = f(t) + \frac{\int_0^T N(t, s, \lambda) f(s) ds}{D(\lambda)},$$

meromorphic for *all*  $\lambda$ .

Requires finite integration domains (compactness).

Hence useful for cylinder kernel only for finite temperature?

Intuitive interpretation: Path integral with vacuum-to-vacuum terms included but then divided out.



## PADÉ APPROXIMANTS

Another  $N/D$  method, defined from formal power series.

Truncations of Fredholm formula are *not* standard Padé approximants; contain nontrivial information about terms that Padé sets to 0.

Padé approximants to operator matrix elements are not bilinear in the state vectors.

# MULTIPLE REFLECTION EXPANSION FOR CYLINDER KERNEL

Although its PDE is not Volterra, let us proceed formally with its Balian–Bloch series.

Experience shows that it seems to sit right on the edge of the circle of convergence. Convergence is improved slightly by taking derivatives to get  $\langle T_{\mu\nu} \rangle$  components.



Starting with quantum, heat, or resolvent kernel does not help. Indeed,

$$T(t) = \frac{1}{\pi} \int_0^\infty e^{-tk} \operatorname{Im} G(\mathbf{r}, \mathbf{r}', k) d(k^2),$$

$$\bar{T}(\tau) = -\frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-\tau^2/4t} K(t) dt.$$

But these formulas involve integration over parameters in limits where the expansions are not valid.

# What Is the Relation between MRE and Optical Approximation?

Recall from last year that the optical approximation fails in higher order and the correct repair is hard to guess as an ansatz. Can we deduce the truth from the MRE?

Look at what Balian and Bloch say:

Ann. Phys. **69** (1972) 76 (“Paper III”)

## ONE-DIMENSIONAL CASE: PRELIMINARIES

Green function for Poisson equation in  $\mathbf{R}^2$  is  
(2D Coulomb potential)

$$G_0(t, x; s, y) = -\frac{1}{4\pi} \ln[(t - s)^2 + (x - y)^2] + C.$$

$$\bar{T}(t, x, y) = -2G(t, x; 0, y), \quad T = -2 \frac{\partial G}{\partial t}.$$

$$\frac{\partial G_0}{\partial x}(t, x; s, y) = -\frac{1}{2\pi} \frac{x - y}{(t - s)^2 + (x - y)^2}, \quad \text{etc.}$$

Recall

$$\lim_{x \downarrow 0} \frac{1}{\pi} \frac{x}{(t-s)^2 + x^2} = \delta(t-s).$$

General ansatz:  $G = G_0 + \gamma$ ,

$\gamma$  is an integral over the boundary involving unknown densities for which one derives integral equations.

Then one hopes to solve those equations by iteration (Neumann series), even when they are not Volterra.

## HALF-LINE WITH NEUMANN ENDPOINT

$$\text{Try } \gamma(t, x; s_0, y_0) = \int_{-\infty}^{\infty} ds G_0(t, x; s, 0) \mu_0(s).$$

As always in the boundary-integral method, we must make a careful distinction between the value of such an integral exactly on the boundary ( $x = 0$ ) and the limit of the integral as the boundary is approached from the interior.

$$\begin{aligned} \frac{\partial \gamma}{\partial x}(t, 0; s_0, y_0) &= - \int_{-\infty}^{\infty} ds \frac{x - y}{2\pi} [(t - s)^2 + (x - y)^2]^{-1} \Big|_{x=y=0} \mu_0(s) \\ &= 0, \end{aligned}$$

but

$$\begin{aligned} \frac{\partial \gamma}{\partial x}(t, x; s_0, y_0) &= - \int_{-\infty}^{\infty} ds \frac{x}{2\pi} [(t-s)^2 + x^2]^{-1} \mu_0(s) \\ &\rightarrow -\frac{1}{2} \mu_0(t) \quad \text{as } x \downarrow 0, \end{aligned}$$

and *the latter* must be chosen to satisfy the Neumann boundary condition

$$0 = \frac{\partial G}{\partial x}(t, 0^+; s_0, y_0) = \frac{\partial G_0}{\partial x} + \frac{\partial \gamma}{\partial x}.$$

You get

$$\mu_0(t) = 2 \frac{\partial G_0}{\partial x}(t, 0^+; s_0, y_0) = \frac{y_0}{\pi} [(t - s_0)^2 + y_0^2]^{-1},$$

and hence

$$\begin{aligned} & \gamma(t, x; s_0, y_0) \\ &= -\frac{y_0}{4\pi^2} \int_{-\infty}^{\infty} ds \ln[(t - s)^2 + x^2] [(s - s_0)^2 + y_0^2]^{-1}. \end{aligned}$$



But from the method of images we know a more elementary formula for  $\gamma$ :

$$\begin{aligned}\gamma(t, x; s_0, y_0) &= G_0(t, x; s_0, -y_0) \\ &= -\frac{1}{4\pi} \ln[(t - s_0)^2 + (x + y_0)^2].\end{aligned}$$

**Lemma:** Let  $x$  and  $y$  be positive. Then (indeed)

$$\int_{-\infty}^{\infty} ds \frac{\ln[(t - s)^2 + x^2]}{(s - r)^2 + y^2} = \frac{\pi}{y_0} \ln[(t - r)^2 + (x + y)^2].$$

## INTERVAL $(0, L)$

Try  $\gamma(t, x; s_0, y_0)$

$$= \int_{-\infty}^{\infty} ds G_0(t, x; s, 0)\mu_0(s) + \int_{-\infty}^{\infty} ds G_0(t, x; s, L)\mu_L(s).$$

BC are  $\frac{\partial \gamma}{\partial n}(0^+) = - \frac{\partial G_0}{\partial x}(t, 0; s_0, y_0),$

$$\frac{\partial \gamma}{\partial n}(L^-) = + \frac{\partial G_0}{\partial x}(t, L; s_0, y_0).$$

We get the basic boundary-integral equations of the system,

$$\begin{aligned}\mu_0(t) &= \frac{1}{\pi} \frac{y_0}{(t - s_0)^2 + y_0^2} \\ &+ \frac{1}{\pi} \int_{-\infty}^{\infty} ds \frac{L}{(t - s)^2 + L^2} \mu_L(s),\end{aligned}$$

$$\begin{aligned}\mu_L(t) &= \frac{1}{\pi} \frac{L - y_0}{(t - s_0)^2 + (L - y_0)^2} \\ &+ \frac{1}{\pi} \int_{-\infty}^{\infty} ds \frac{L}{(t - s)^2 + L^2} \mu_0(s).\end{aligned}$$

More abstractly, fix a vertex  $v_*$ ; then

$$\mu_{v_*}(t) = 2 \frac{\partial G_0}{\partial n}(t, v_*; s_0, y_0) + 2 \int_{\partial \tilde{\Gamma}} \frac{\partial G_0}{\partial n}(t, v_*; s, v) \mu_v(s),$$

where a sum over  $v$  is implicit in integration over  $\partial \tilde{\Gamma} = \mathbf{R} \times \partial \Gamma$ , and

$$\frac{\partial G_0}{\partial n}(t, v_*; s, v_*) = 0.$$

Even more abstractly, the integral equation has the form

$$\mu = g_0 + K\mu,$$

so that formally

$$\mu = (1 - K)^{-1}g_0 \sim (1 + K + K^2 + \dots)g_0 .$$

The operator has norm 1, so the convergence is conditional (for derivatives of  $\bar{T}$ ) and only distributional for  $\bar{T}$  itself (for test functions orthogonal to constants). Nevertheless, iterating yields order by order the image solution!

## QUANTUM GRAPHS

Half-line case generalizes easily to any infinite star graph (giving the known solution).

Interval case generalizes with difficulty to any compact graph. The system of integral equations is complicated and redundant, and it is not obvious that the relevant operator has norm 1. Presumably iteration will yield the Wilson solution in principle.

## MULTIDIMENSIONAL CASE

$$G_0(t, \mathbf{x}; s, \mathbf{y}) = C_n [(t - s)^2 + |\mathbf{x} - \mathbf{y}|^2]^{-(n-2)/2}.$$

Try  $G = G_0 + \gamma$  where (Dirichlet case)

$$\gamma(t, \mathbf{x}; s_0, \mathbf{y}_0) = \int_{-\infty}^{\infty} ds \int_S d\sigma_y \frac{\partial G_0}{\partial n_y}(t, \mathbf{x}; s, \mathbf{y}) \mu(s, \mathbf{y}).$$

$\mu$  depends on  $(s_0, \mathbf{y}_0)$ ;  $S = \partial\Omega$ ;  $\hat{\mathbf{n}}$  = inward normal.

As in Balian–Bloch or any PDE book on potential theory,

$$\begin{aligned}\gamma(t, \mathbf{x}^+; s_0, \mathbf{y}_0) &= \int_{-\infty}^{\infty} ds \int_S d\sigma_y \frac{\partial G_0}{\partial n_y}(t, \mathbf{x}; s, \mathbf{y}) \mu(s, \mathbf{y}) \\ &\quad + \frac{1}{2} \mu(t, \mathbf{x}).\end{aligned}$$

So you get the basic boundary integral equation

$$\begin{aligned}\mu(t, \mathbf{x}) &= -2G_0(t, \mathbf{x}; s_0, \mathbf{y}_0) \\ &\quad - 2 \int_{-\infty}^{\infty} ds \int_S d\sigma_y \frac{\partial G_0}{\partial n_y}(t, \mathbf{x}; s, \mathbf{y}) \mu(s, \mathbf{y}).\end{aligned}$$



In lowest order, for  $d = 3$ , I get after several more steps

$$\gamma_1 = \frac{-1}{(2\pi)^3} \int_S d\sigma_y \cos \phi \left[ \frac{l}{r_1^2 r_2 (t^2 + l^2)} + \frac{l^2 - t^2}{r_1 r_2 (t^2 + l^2)^2} \right]$$

— same thing Liu gets in his thesis by Laplace-transforming the Balian–Bloch formula for the first-order term of the spectral density.  $\cos \phi \equiv \hat{\mathbf{n}}_y \cdot \widehat{(\mathbf{x} - \mathbf{y})}$ ;

$$r_1 \equiv |\mathbf{x} - \mathbf{y}|; \quad r_2 \equiv |\mathbf{y} - \mathbf{y}_0|; \quad l \equiv r_1 + r_2.$$

No stationary-phase or semiclassical approximation has been made here — just the leading approximation in the multiple-scattering expansion.

So far the *spatial* boundary is curved (but smooth). If it's flat, should be able to integrate over  $\mathbf{y}$  to get the simple image term as B&B predicted. But the integrals are hard.

## CONCLUSIONS AND PROSPECTS

- Things turn out consistent when they can be calculated (or proved).
  - Those calculations can be surprisingly hard.
  - Direct cylinder calculation has no great advantage (so far) over spectral calculation.
  - Convergence is marginal at best.
- Empirics outrun proofs.
  - Apparently optical approximations become exact for flat surfaces and good for gently curved ones.

- More work is needed to ground optical and diffractive approximations on exact integral-equation formulations.