Closed Path Approach to Casimir Effect in Rectangular Cavities and Pistons

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Outline

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2. Scalar Rectangular Cavity and Piston
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Cylinder kernel

The cylinder kernel is defined as:

$$\overline{T}(t,\mathbf{r},\mathbf{r}') = -\sum \frac{1}{\omega_n}\phi_n(\mathbf{r})\phi_n^*(\mathbf{r}')e^{-t\omega}$$

The exponential ultraviolet cutoff term $e^{-t\omega}$ regularizes the divergent sum and bring a finite energy density.

The cylinder kernel in free space is:

$$\overline{T}^f(t,\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi^2}\frac{1}{t^2+|\mathbf{r}-\mathbf{r}'|^2}$$

The relation between energy density and cylinder kernel:

$$\langle T_{00}\rangle = -\frac{1}{2}\frac{\partial^2}{\partial t^2}\overline{T}(t,\mathbf{r},\mathbf{r})|_{t\to 0} = \frac{1}{2}\sum \omega_n|\phi_n(\mathbf{r})|^2 e^{-t\omega}$$

Method of images

For parallel plates at $x = 0, a$

- Cylinder Kernel satisfies Dirichlet B.C. on both plates

$$\overline{T}_{DD} = \overline{T}^f + D_0\overline{T}^f + D_a\overline{T}^f + D_0 D_a\overline{T}^f + D_a D_0\overline{T}^f + \ldots$$

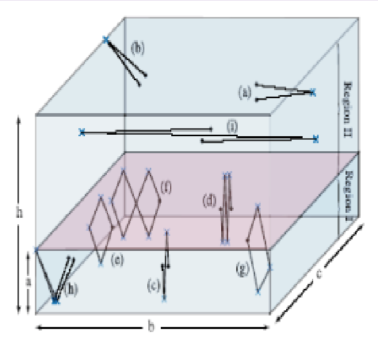
- where the operator D_a is

$$D_a f(x) = -f(2a - x)$$

For rectangular cavity

Cylinder Kernel satisfies Dirichlet B.C. on all faces

$$\begin{aligned}\overline{T}^{DDD} &= D^{eee}\overline{T}^f + D^{oee}\overline{T}^f + D^{eoe}\overline{T}^f + D^{eeo}\overline{T}^f \\ &\quad + D^{ooe}\overline{T}^f + D^{eoo}\overline{T}^f + D^{oeo}\overline{T}^f + D^{ooo}\overline{T}^f \\ &= V^P - V^{S_x} - V^{S_y} - V^{S_z} + V^{E_{xy}} + V^{E_{yz}} + V^{E_{zx}} - V^C\end{aligned} \tag{1}$$



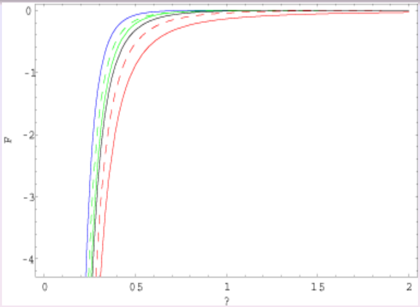
1. Periodic Paths: (c) (d) (i)
   [even][even][even]
2. Side Paths: (a) (e) (f)
   [odd][even][even], [even][odd][even], [even][even][odd]
3. Edge Paths: (b) (g)
   [odd][odd][even], [even][odd][odd], [odd][even][odd],
4. Corner Paths: (h)
   [odd][odd][odd]

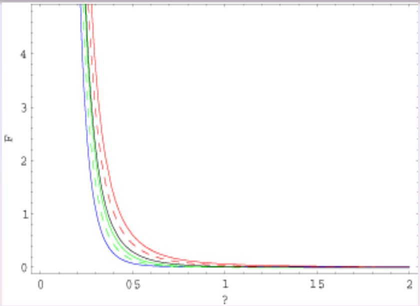
$$E^{\alpha\beta\gamma}_{piston} = E^{\alpha\beta\gamma}_{a,b,c} + E^{\alpha\beta\gamma}_{L-a,b,c}|_{L\to\infty}$$

$$F^{\alpha\beta\gamma}_{piston} = F^{\alpha\beta\gamma}_{a,b,c} + F^{\alpha\beta\gamma}_{L-a,b,c}|_{L\to\infty}$$

$$= F^{P}_{1} + \eta_\beta F^{S_y}_{1} + \eta_\gamma F^{S_z}_{1} + \eta_\beta\eta_\gamma F^{E_{yz}}_{1} \tag{2}$$

where $\alpha, \beta, \gamma = D, N, M$ and
$\eta_D = -1, \eta_N = 1, \eta_M = 1$





1. Solid red= F^{MNN}, dashed red= F^{MMN}, solid black= F^{MMM}, solid green= F^{MDN}, dashed green= F^{MMD} and solid blue= F^{MDD}.
2. As $\eta \to 0$, all piston forces reduce to parallel plates; while as $\eta \to \infty$, all piston forces go to 0.
3. When B.C. at $x = 0, a$ are both Dirichlet or both Neumann, the piston forces will be always attractive no matter what B.C. are on other sides. when mixed B.C. are at $x = 0, a$, the piston force will be always repulsive.
4. $| F^{P}_{piston} | > | F^{S_y}_{piston} | > | F^{E_{yz}}_{piston} |$, F^{P}_{piston} are dominant.

Hertz potential

- EM field can be represented by the 4-vector $(\Phi, \mathbf{A})$, **E** and **B** can then be expressed as

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \partial_t \mathbf{A} \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \tag{3}$$

- However, that is not the only way to express EM field. Define Herta potential as below

$$(\Phi, \mathbf{A}) = (-\nabla \cdot \Pi_e, \partial_t \Pi_e + \nabla \times \Pi_m) \tag{4}$$

- For a highly symmetric geometry such as rectangular cavity, we can choose the Hertz potentials as $\Pi_e = \varphi \vec{e}_3$ and $\Pi_m = \psi \vec{e}_3$.

$$\begin{aligned} (\Phi, \mathbf{A}) &= (-\partial_3\phi, \partial_2\psi, -\partial_1\psi, \partial_0\varphi) \\ \mathbf{E} &= -\nabla\Phi - \partial_t\mathbf{A} = (\partial_1\partial_3\phi - \partial_0\partial_2\psi, \partial_2\partial_3\phi + \partial_0\partial_1\psi, \partial_3^2\phi - \partial_0^2\phi) \\ \mathbf{B} &= \nabla \times \mathbf{A} = (\partial_0\partial_2\phi - \partial_1\partial_3\psi, -\partial_0\partial_1\phi + \partial_2\partial_3\psi, \partial_3^2\psi - \partial_0^2\psi) \end{aligned} \tag{5}$$

EM cavity with all faces conducting

- There are 2 kinds of B.C. for EM field:
  1. Conducting B.C. (CBC): $E_t = 0, B_n = 0$ on the boundary
  2. Permeable B.C. (PBC): $E_n = 0, B_t = 0$ on the boundary.
- B.C. of **E** and **B** $\Longrightarrow$ B.C. of Hertz potential ϕ and ψ

$$\begin{aligned} B_x|_{x=0,a} &= 0 \qquad E_y = E_z|_{x=0,a} = 0 \\ B_y|_{y=0,b} &= 0 \qquad E_x = E_z|_{y=0,b} = 0 \\ B_z|_{z=0,c} &= 0 \qquad E_x = E_y|_{z=0,c} = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} \phi_x|_{x=0,a} &= 0 \qquad \partial_x\psi_x|_{x=0,a} = 0 \\ \phi_y|_{y=0,b} &= 0 \qquad \partial_y\psi_y|_{y=0,b} = 0 \\ \partial_z\phi_z|_{z=0,c} &= 0 \qquad \psi_z|_{z=0,c} = 0 \end{aligned} \tag{7}$$

- The 2 Hertz potential are $\phi = \phi^{DDN}$ and $\psi = \psi^{NND}$
- $E^{CCC}_{cavity} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) = E^{DDN}_{cavity} + E^{NND}_{cavity}$

Boundary condition equivalence

- If we replace CBC by PBC and replace PBC by CBC, then the two scalar Hertz potentials exchange to each other $(\phi, \psi) \to (\psi, \phi)$, which makes **E** and **B** exchange their values but keeps the energy density $\varepsilon = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ unchanged.
- A straightforward conclusion would be $E^{CCC}_{cavity} = E^{PPP}_{cavity}$ and therefore $F^{CCC}_{piston} = F^{PPP}_{piston}$.
- Similar as scalar piston, all *a*-dependent terms (if appearing) are: F^P, $\eta_\beta F^{S_y}$, $\eta_\gamma F^{S_z}$, $\eta_\beta \eta_\gamma F^{E_{yz}}$ it has nothing to do with η_α, so B.C. at $x = 0$, a changing from CBC to PBC won't change the piston force, $F^{C\beta\gamma}_{piston} = F^{P\beta\gamma}_{piston}$.
- Therefore $F^{CCC} = F^{PCC}$ and $F^{PPP} = F^{CPP}$
- So far, without any calculation, we know that $F^{CCC} = F^{PPP} = F^{CPP} = F^{PCC} = F^{DDN} + F^{DND}$

Translate EM piston to Scalar Pistons

- It turns out that a EM piston force can be always translated to 2 scalar piston forces, determined by the Hertz potential.
- $$F^{CCC} = F_1^{DDN} + F_1^{DND}$$
  $$F^{CPC} = F_1^{DDD} + F_1^{NNN}$$
  $$F^{CMC} = F_2^{NMN} + F_2^{DMD}$$
  $$F^{CMM} = 2F_3^{NMM}$$
  $$F^{MCC} = F_2^{MDN} + F_2^{MND}$$
  $$F^{MPC} = F_2^{MDD} + F_2^{MNN}$$
  $$F^{MMC} = F_3^{MMN} + F_3^{MMD}$$
  $$F^{MMM} = 2F_4^{MMM}$$
- The piston force of EM piston is closely related to the piston forces of 2 corresponding scalar piston by use of the Hertz potentials.
- Cylinder Kernel$\Longrightarrow$ Energy Density$\Longrightarrow$ Total Energy$\Longrightarrow$ Casimir Force of a cavity$\Longrightarrow$ Casimir force of a piston