Semiclassical Corrections to the Mass of a Black Hole coupled to Scalars (M. Schaden, T. Jain, TAMU09)

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Concepts

- The spectrum of the Laplace-Beltrami operator and therefore also the zero-point energy ‘ $\frac{\hbar}{2}\sum \omega_n[g^{\mu\nu}]$ ‘ of a (scalar) quantum field depend on the background metric. (Casimir effect).
- The zero-point energy diverges, but its change with certain variations in the background metric can (and physically ought to be) finite – for instance for an Extremal Reissner-Nordström (E-RN) background with conformally coupled scalars (classical BBMB-solution).
- Change in zero-point energy of quantum fields enters the relation between mass and radius of the BH. We compute the leading semi-classical (WKB) contribution to this zero-point energy due to periodic classical orbits .
- To enhance the importance of the zero-point energy consider N identical conformally coupled scalars – formally avoids complications with GR.

The Setup

I will consider the change in vacuum (zero-point) energy due to the formation of a black hole. What should the black hole be made of? Dust? Quantum fields?

We should expect to encounter terrible divergences unless we follow a "physically plausible ", if idealized scenario. I.e., if the BH forms from dust, we would have to address (renormalize) some of the properties of this material--too complicated.

Simplest consistent scenario: a black hole is formed by scalar fields free scalars ->BH+bound scalar

Indeed: A self-consistent BH –solution of classical GR was found in the 1970's : Bochorova-Bronnikov-Melnikov-Bekenstein black hole

Reissner-Nordström BH

The Reissner-Nordström metric:

$$ds^2 = v(r)c^2dt^2 - dr^2 / v(r) - r^2 d\Omega^2, \quad v(r) = (1 - \frac{R_+}{r})(1 - \frac{R_-}{r}),$$

For general $R_+ > R_-$ is a static and spherically symmetric vacuum solution of the Einstein-Maxwell equations. It describes a BH of mass,

$$M_{BH}c^2 = \frac{4\pi}{\kappa}(R_+ + R_-) = E_{BH},$$

vanishing angular momentum and electric charge

$$Q^2 = \frac{8\pi}{\kappa} R_+ R_- ; \qquad \text{electric field} \quad \vec{\mathcal{E}} = \hat{r}\phi_{,r} = \hat{r}\frac{Q}{r^2}$$

Classical BH Thermodynamics

Bekenstein and Hawking interpreted changes in the (classical) mass of a BH thermodynamically,

$$dE_{BH} = \frac{4\pi}{\kappa}(dR_+ + dR_-) = TdS + \Phi dQ$$

With BH-entropy: $S = \frac{\pi k}{\hbar c \kappa} A_{BH} = \frac{4\pi^2 k}{\hbar c \kappa} R_+^2$

Temperature: $T = \frac{\hbar c}{4\pi k} \frac{R_+ - R_-}{R_+^2}$

Potential: $\Phi = \frac{Q}{R_+} = \sqrt{\frac{8\pi R_-}{\kappa R_+}}$

No Hawking radiation from E-RN

$$R_+ = R_- = R$$

Note: Although T and S depend on $\hbar$, the “free energy” is entirely classical ! Demers, Lafrance, Myers 1995: Free energy of E-RN is not renormalized by scalars . Why?

GR coupled to N (massless) scalars

$$\Gamma^{(0)} = \frac{1}{2}\int d^4x\sqrt{-g}\left[\frac{R}{\kappa} - g^{\mu\nu}\vec{\phi}_{,\mu}\vec{\phi}_{,\nu} - \xi R\vec{\phi}^2\right]$$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa\, T_{\mu\nu} \quad \wedge \quad \Box_g \vec{\phi} = \xi R \vec{\phi}$$

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} + \xi[g_{\mu\nu}\Box_g \phi^2 - \phi^2_{;\mu\nu} + G_{\mu\nu}\phi^2]$$

Global SO(N)-symmetry : consider classical solution for which all but one, ϕ , of the N scalars vanish:

$\xi = 0$ (minimal) $\Rightarrow$ Schwarzschild: $R_- = 0$, ϕ=const.

$\xi = \frac{1}{6}$ (conformal) $\Rightarrow$ Extremal RN: $R_- = R_+ = R$, $\phi = \sqrt{\frac{6}{\kappa}}\frac{R}{r-R}$

The E-RN solves GR with conformal coupled SCALAR= (Bochorova-Bronnikov-Melnikov-Bekenstein) black hole.(1970's)

1-loop effective action

$$\Gamma^{(1)} = \Gamma^{(0)} + \hbar\tilde{N}\ln\overline{\mathrm{Det}}^{1/2}[\Box_g] + \text{ct's} \qquad \text{with } \tilde{N} = N+2,$$

$$\text{ct's} = \int d^4x\sqrt{-g}\left(Z_\Lambda + Z_R R + Z_{R^2}\left(\tfrac{5}{2}R^2 - R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right)$$

$\Rightarrow$ No logarithmic 1-loop divergence for backgrounds with

$$\int d^4x\sqrt{-g}\left(\tfrac{5}{2}R^2 - R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) = 0!$$

Makes sense? to compute (finite) Casimir contributions to the mass of a BBMB black hole, e.g.

$$M_{E-RN}c^2 = \frac{8\pi}{\kappa}R + \chi\frac{\tilde{N}\hbar c}{R} \qquad \text{Q: } \chi \gtrless 0 \text{ ?}$$

Partial Wave Analysis

$$\ln \overline{\mathrm{Det}}^{1/2}[\Box_R/\Box_0] = \tfrac{1}{2}\mathrm{Tr}\,\ln[\Box_R/\Box_0] = \tfrac{-i}{2}\int d^4x \int_0^\infty d\lambda\, \tilde{G}(x,x;\lambda)$$

$$\tilde{G}(x,y;\lambda) = i\left\langle y\right|\frac{1}{\lambda+\Box_R} - \frac{1}{\lambda+\Box_0}\left|x\right\rangle = G_R(x,y;\lambda) - G_0(x,y;\lambda) \quad \text{with} \quad \left\langle y\middle|x\right\rangle = \delta^4(x-y)$$

$PWA:$

$$\tilde{G}(x,x';\lambda) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\Omega')Y^*_{lm}(\Omega)\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\tilde{G}^{(l)}(r,r';\lambda,\varepsilon)\exp[i\varepsilon c(t'-t)]$$

$$\hbar \ln \overline{\mathrm{Det}}^{1/2}[\Box_R/\Box_0] = t\,\frac{-i\hbar c}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\int_0^\infty d\varepsilon\int_0^\infty e^{2\pi i n l}\,l\,dl\int_0^\infty r^2dr\int_0^\infty d\lambda\,\tilde{G}^{(l)}(r,r;\lambda,\varepsilon)$$

Where the Poisson identity

$$\sum_{l=0}^{\infty}(2l+1)f(l(l+1)) = \sum_{n=-\infty}^{\infty}(-1)^n\int_{\cancel{-1/4}}^{\infty} e^{2\pi i n\sqrt{l^2\cancel{+1/4}}}f(l^2)\,dl^2 \qquad \text{was used.}$$

Proceed with WKB-approximation

Semiclassics…. The radial-Green's functions satisfy:

$$(\lambda - \frac{\varepsilon^2}{v(r)} + \frac{l^2}{r^2} - \frac{1}{r^2}\partial_r r^2 v(r)\partial_r)G^{(l)}(r,r';\lambda,\varepsilon) = \frac{i\delta(r-r')}{r^2}$$

Writing: $G^{(l)}(r,r';\lambda,\varepsilon) = A^{(l)}(r,r';\lambda,\varepsilon)\exp[iS^{(l)}(r,r';\lambda,\varepsilon)]$

1) $\lambda = \frac{\varepsilon^2}{v(r)} - \frac{l^2}{r^2} - v(r)(\partial_r S^{(l)})^2 + \frac{1}{A^{(l)}r^2}\partial_r r^2 v(r)\partial_r A^{(l)}$

2) $A^{(l)}\delta(r-r') = -\partial_r (A^{(l)})^2 r^2 v(r)(\partial_r S^{(l)}))$

WKB-approx:
=> HJ-equ.

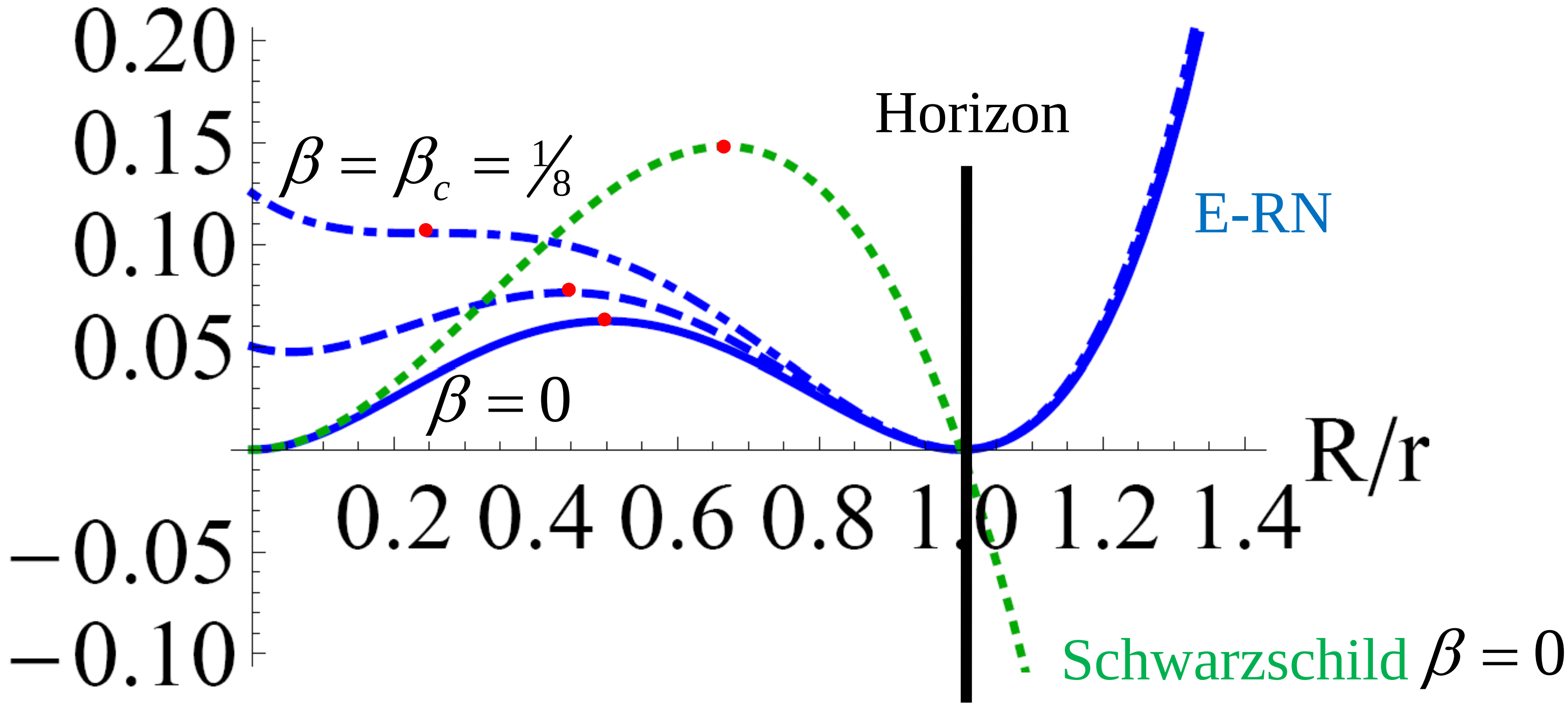
$$G^{(l)}(r,r;\lambda,\varepsilon) \sim \frac{1}{2r^2 k(r;\varepsilon,l,\lambda)}\exp\left[i\oint_r \frac{dz}{v(z)}k(z;\varepsilon,l,\lambda)\right]$$

$$k(r;\varepsilon,l,\lambda) = \sqrt{\varepsilon^2 - \left(\frac{l^2}{r^2} + \lambda\right)v(r)}$$

$$k^2(r;\varepsilon,l,\lambda)=\varepsilon^2-\left(\frac{l^2}{r^2}+\lambda\right)v(r)=\frac{l^2}{R^2}[\alpha^2-V_{eff}(x=\frac{R}{r};\beta)]$$

with $\alpha=\frac{\varepsilon R}{l},\beta=\frac{\lambda R^2}{l^2}$

$V_{eff}(x)=v(x)(x^2+\beta)$ • =Unstable circular orbit

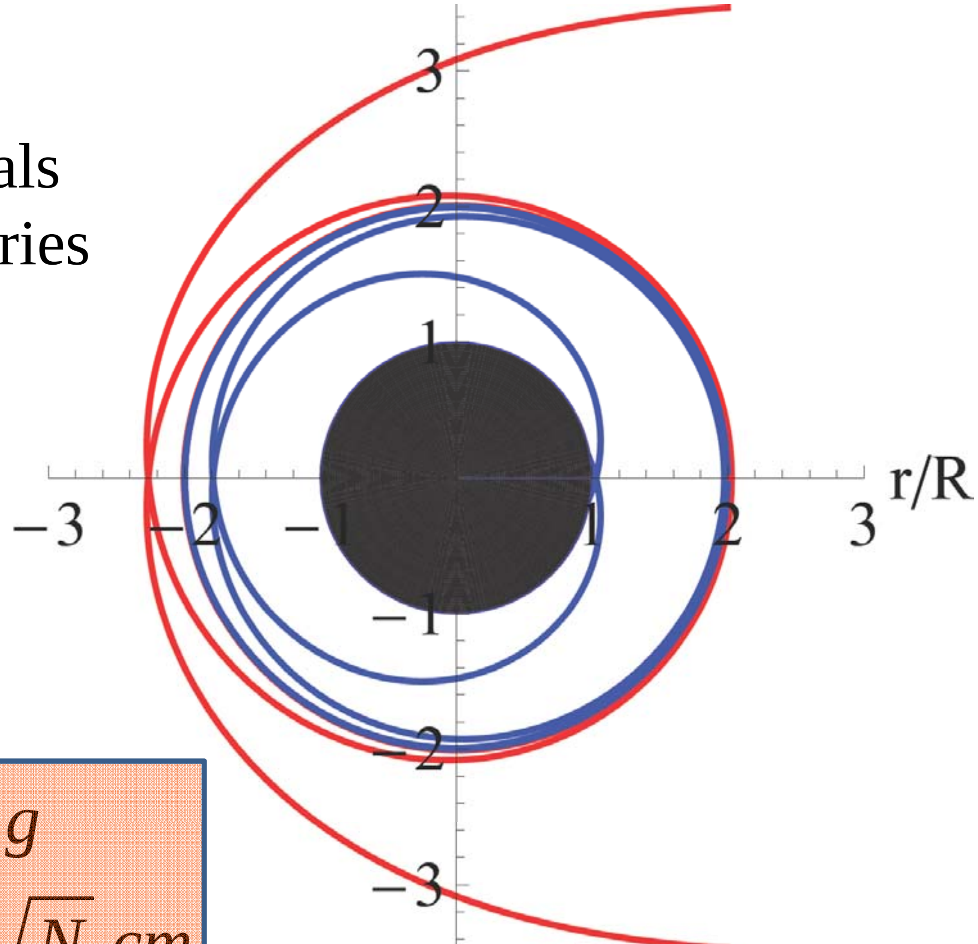


$$\mathcal{E}_{Cas} \sim \frac{-i\hbar c}{2\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \int_0^\infty d\varepsilon \int_0^\infty \ell d\ell \int_0^\infty dr \int_0^\infty d\lambda \frac{e^{2i\pi n\ell + 2i\int_{\bar{r}}^{r} \frac{dz}{v(z)} k(z;\varepsilon,\ell,\lambda)}}{k(r;\varepsilon,\ell,\lambda)}$$

Saddlepoint evaluation of integrals about (unstable) critical trajectories with $\lambda = 0, r = 2R, l = 2\varepsilon R$ gives:

$$\chi = +0.003$$

(preliminary)



The smallest E-RN black hole:

$$M_{\min} \sim 0.05 m_P \sqrt{N} \sim 1.2\sqrt{N}\ \mu g$$

$$R_{\min} \sim 0.05 \ell_P \sqrt{N} \sim 8.9\times 10^{-35}\sqrt{N}\ cm$$

Conclusion and Speculation

- Zero-point energy tends to increase the mass (energy) of small black holes and may prevent them from forming (much) below Planck scales.
- The smallest mass E-RN black hole, here estimated semiclassically, compares well (qualitatively and quantitatively) with a Loop Quantum Gravity approach by L. Modesto (ArXive:0811.2196) who finds that a ~ $0.1\, m_P$ Schwarzschild black hole is stabilized by graviton fluctuations ($\tilde{N} = 2$) .
- The accuracy of a semiclassical calculation could be doubted for such tiny BH. But dimensional analysis determines the form of the quantum correction to the BH mass and the dimensionless coefficient may be estimated for large R (or N), where semiclassics generally would be trusted.
- Could $M \ll \mu g$ BH "grow " by emitting negative energy Hawking radiation thereby lowering the zero-point energy of quantum fields….? Is Dark Matter composed of stable primordial μg -BHs ?