Casimir Self-Energies of Triangular Cylinders

(Work in progress)

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NSF, DOE support

Outline

- Self-Energy
- Methodology
  - Mode Summation
  - Regularization Methods
- Equilateral Triangular Cylinder
- Hemiequilateral Triangular Cylinder
- Square Cylinder
- Isosceles Right Triangular Cylinder

Self-Energy

- Parallel plates
  - Casimir, 1948
- Conducting sphere
  - Boyer, 1969
- Cylinders
  - Conducting circular cylinder
    - DeRaad & Milton, 1981
  - Square cylinder
    - Lukosz, 1973
- Triangular geometries (Inui; Ahmedov & Duru)

Methodology: Mode Summation

- Infinite cylinder

$$\mathcal{E} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{m,n}\sqrt{k^2+\gamma_{mn}^2}$$

- Eigenvalues

$$\left(-\nabla_\perp^2 + \gamma_{mn}^2\right)\Phi_{mn}(\mathbf{r}_\perp) = 0$$

$$\Phi(\mathbf{r}_\perp)|_C = 0 \qquad \partial_n\Phi(\mathbf{r}_\perp)|_C = 0$$

Divergences, Singularities, Poles,...

- (www.tsa.gov)

Explosive & Flammable Materials, Disabling Chemicals & Other Dangerous Items

Explosive Materials	**Carry-on**	**Checked**
Blasting Caps	No	No
Dynamite	No	No
Fireworks	No	No
Flares (in any form)	No	No
Hand Grenades	No	No
Plastic Explosives	No	No
Realistic Replicas of Explosives	No	No
Flammable Items	**Carry-on**	**Checked**
Aerosol (any except for personal care or toiletries in limited quantities)	No	No
Fuels (including cooking fuels and any flammable liquid fuel)	No	No
Gasoline	No	No
Gas Torches	No	No
Lighter Fluid	No	No
Common Lighters - Lighters without fuel are permitted in checked baggage. Lighters with fuel are prohibited in checked baggage, unless they adhere to		

1/0 , log(0) , ...

Methodology: Regularization Methods

1/ Dimensional Regularization

$$dk \to d^d k$$

$$\mathcal{E} = \frac{1}{2} \lim_{d \to 1} (4\pi)^{-d/2} \frac{\Gamma(-(1+d)/2)}{\Gamma(-1/2)} \sum_{m,n} \left(\gamma_{mn}^2\right)^{(1+d)/2}$$

Chowla-Selberg Summation Formula

$$\sum_{m,n}{}' (m^2 + a\,mn + b\,n^2)^{-s} = 2\zeta(2s) + \frac{2^{2s}\sqrt{\pi}}{\Gamma(s)\Delta^{s-1/2}} \zeta(2s-1)\Gamma(s-1/2)$$
$$+ \frac{2^{s+3/2}\pi^s}{\Gamma(s)\Delta^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sigma_{1-2s}(n) \cos(n\pi a)\, 2K_{1/2-s}(n\pi\sqrt{\Delta})$$

$$\Delta = 4b - a^2 > 0 \qquad \sigma_k(n) \equiv \sum_{d|n} d^k$$

Methodology: Regularization Methods

2/ Cutoff Regularization

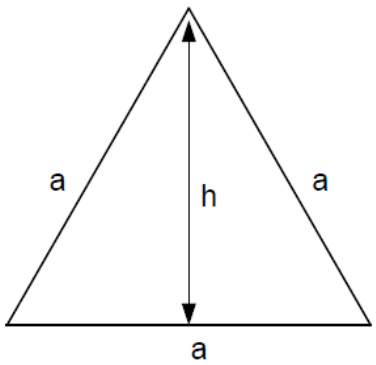
$$\mathcal{E} = \frac{1}{2} \lim_{\tau \to 0} \left(-\frac{d}{d\tau} \right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{m,n} e^{-\tau \sqrt{k^2 + \gamma_{mn}^2}}$$

Poisson's Summation Formula

$$\sum_{p=-\infty}^{\infty} f(p) = \sum_{q=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} dp \, e^{2\pi i p \, q} f(p) \right)$$

Equilateral Triangular Cylinder

1/ Dirichlet B.C. $\Phi(\mathbf{r}_\perp)|_C = 0$



$$\gamma_{mnp}^2 = \frac{2\pi^2}{3h^2}(m^2 + n^2 + p^2)$$

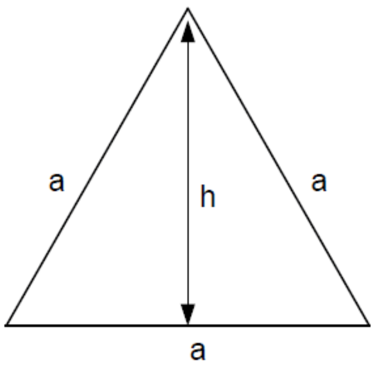
$$m + n + p = 0$$

$$\mathcal{E}_{Eq}^{(D)} = \frac{1}{144\pi^2 h^2}\left(8\pi\zeta(3) - 3^{3/2}\zeta(4) - 4(12)^{3/4}\sum_{n=1}^{\infty} n^{-3/2}(-1)^n \sigma_3(n) K_{3/2}(n\pi\sqrt{3})\right)$$

$$\mathcal{E}_{Eq}^{(D)} = \frac{1}{144\pi^2 h^2}\left(12\pi\zeta(3) - \frac{10\sqrt{3}}{3}\zeta(4) - \frac{16\sqrt{3}}{3}\sum_{m,n=1}^{\infty}\frac{1+(-1)^{m+n}}{(m^2+n^2/3)^2}\right)$$

$$\mathcal{E}_{Eq}^{(D)} = +\frac{0.0177891}{h^2}$$

Equilateral Triangular Cylinder



2/ Neumann B.C. $\partial_n \Phi(\mathbf{r}_\perp)|_C = 0$

$$\gamma_{mnp}^2 = \frac{2\pi^2}{3h^2}(m^2 + n^2 + p^2)$$

$$m + n + p = 0$$

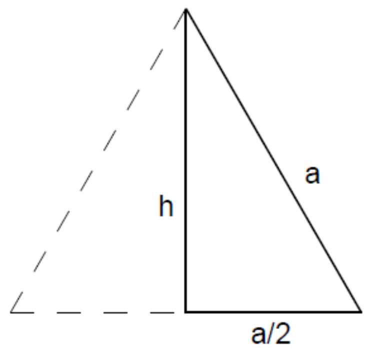
$$\mathcal{E}_{Eq}^{(N)} = \mathcal{E}_{Eq}^{(D)} - \frac{\zeta(3)}{6\pi h^2}$$

$$\mathcal{E}_{Eq}^{(N)} = -\frac{0.045982}{h^2}$$

3/ E&M

$$\mathcal{E}_{Eq}^{(EM)} = -\frac{0.028193}{h^2}$$

Hemiequilateral Triangular Cylinder



1/ Dirichlet B.C.

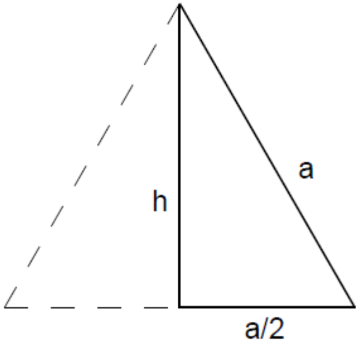
$$\gamma_{mnp}^2 = \frac{2\pi^2}{3h^2}(m^2 + n^2 + p^2)$$

$$m + n + p = 0$$

$$\mathcal{E}_{Hemi}^{(D)} = \frac{\mathcal{E}_{Eq}^{(D)}}{2} + \frac{\zeta(3)}{8\pi h^2}$$

$$\mathcal{E}_{Hemi}^{(D)} = +\frac{0.0567229}{h^2}$$

Hemiequilateral Triangular Cylinder



2/ Neumann B.C.

$$\gamma_{mnp}^2 = \frac{2\pi^2}{3h^2}(m^2 + n^2 + p^2)$$
$$m + n + p = 0$$

$$\mathcal{E}_{Hemi}^{(N)} = \frac{\mathcal{E}_{Eq}^{(N)}}{2} - \frac{\zeta(3)}{8\pi h^2}$$

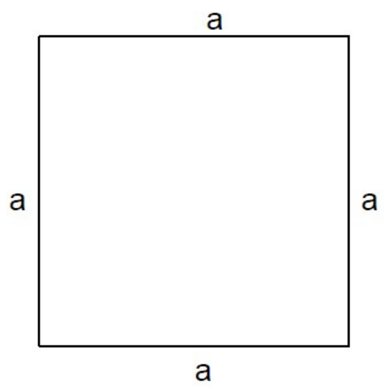
$$\mathcal{E}_{Hemi}^{(N)} = -\frac{0.0708193}{h^2}$$

3/ E&M

$$\mathcal{E}_{Hemi}^{(EM)} = -\frac{0.014096}{h^2}$$

Square Cylinder

1/ Dirichlet B.C.



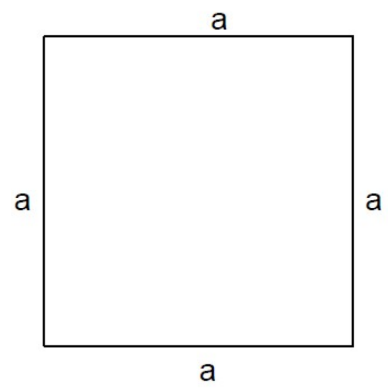
$$\gamma_{mn}^2 = \frac{\pi^2}{a^2}(m^2 + n^2)$$

$$\mathcal{E}_{Sq}^{(D)} = \frac{1}{32\pi^2 a^2}\left(\pi\zeta(3) - 2\zeta(4) - 8\pi^2 \sum_{n=1}^{\infty} n^{-3/2}\sigma_3(n) K_{3/2}(2n\pi)\right)$$

$$\mathcal{E}_{Sq}^{(D)} = \frac{1}{32\pi^2 a^2}\left(2\pi\zeta(3) - 4\zeta(4) - 4\sum_{m,n=1}^{\infty}(m^2+n^2)^{-2}\right)$$

$$\mathcal{E}_{Sq}^{(D)} = +\frac{0.00483155}{a^2}$$

Square Cylinder



2/ Neumann B.C.

$$\gamma_{mn}^2 = \frac{\pi^2}{a^2}(m^2 + n^2)$$

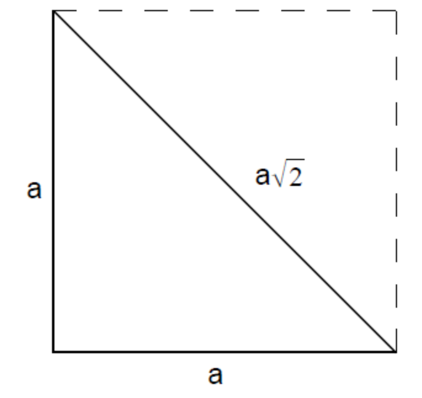
$$\mathcal{E}_{Sq}^{(N)} = \mathcal{E}_{Sq}^{(D)} - \frac{\zeta(3)}{8\pi a^2}$$

$$\mathcal{E}_{Sq}^{(N)} = -\frac{0.042996}{a^2}$$

3/ E&M

$$\mathcal{E}_{Sq}^{(EM)} = -\frac{0.0381645}{a^2}$$

Right Isosceles Triangular Cylinder



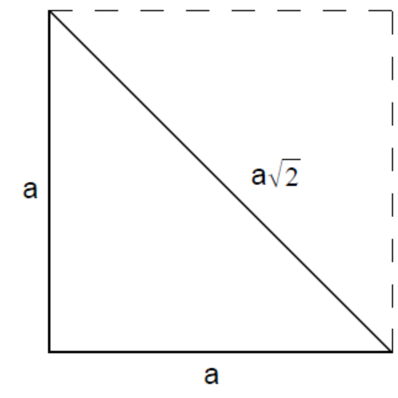
1/ Dirichlet B.C.

$$\gamma_{mn}^2 = \frac{\pi^2}{a^2}(m^2 + n^2)$$

$$\mathcal{E}_{Iso}^{(D)} = \frac{\mathcal{E}_{Sq}^{(D)}}{2} + \frac{\zeta(3)}{16\pi a^2}$$

$$\mathcal{E}_{Iso}^{(D)} = +\frac{0.026399}{a^2}$$

Right Isosceles Triangular Cylinder



2/ Neumann B.C.

$$\gamma_{mn}^2 = \frac{\pi^2}{a^2}(m^2 + n^2)$$

$$\mathcal{E}_{Iso}^{(N)} = \frac{\mathcal{E}_{Sq}^{(N)}}{2} - \frac{\zeta(3)}{16\pi a^2}$$

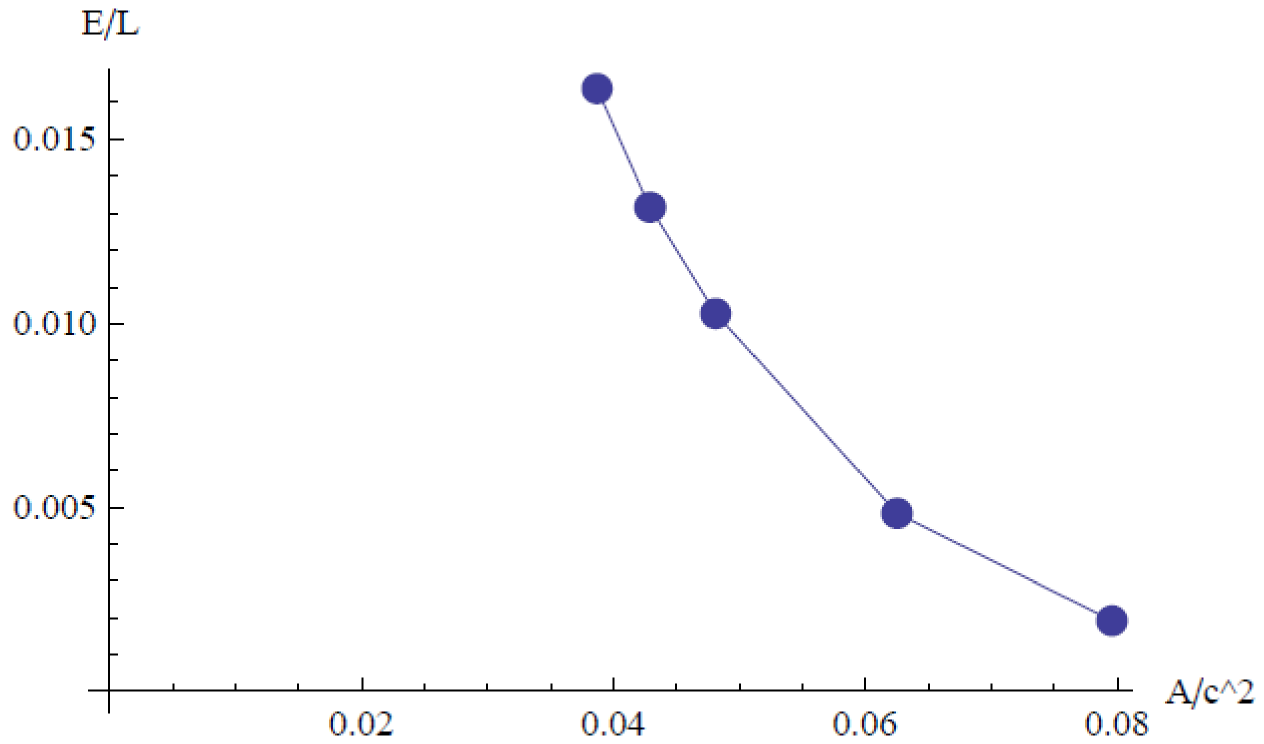
$$\mathcal{E}_{Iso}^{(N)} = -\frac{0.0454125}{a^2}$$

3/ E&M

$$\mathcal{E}_{Iso}^{(EM)} = -\frac{0.019014}{h^2}$$

Energy density plot

- Dirichlet



Conclusions

- Casimir energies of triangular cylinders
- Consistent regularization results
- Self-energy?

Acknowledgements

Rectangular Cylinder

1/ Dirichlet B.C.

$$\gamma_{mn}^2 = \pi^2((m/a)^2 + (n/b)^2)$$

$$\mathcal{E}_{Rect}^{(D)} = \frac{1}{32\pi^2 a^2}\left((a/b)^2\pi\zeta(3) - 2(a/b)^3\zeta(4) - 8\pi^2(a/b)^{3/2}\sum_{n=1}^{\infty} n^{-3/2}\sigma_3(n)K_{3/2}(2n\pi a/b)\right)$$

$$\mathcal{E}_{Sq}^{(D)} = \frac{1}{32\pi^2 a^2}\left((1 + (a/b)^2)\pi\zeta(3) - 2((a/b)^3 + b/a)\zeta(4) - 4(a/b)^3\sum_{m,n=1}^{\infty}(m^2 + (n\,a/b)^2)^{-2}\right)$$

Rectangular Cylinder

2/ Neumann B.C.

$$\gamma_{mn}^2 = \pi^2((m/a)^2 + (n/b)^2)$$

$$\mathcal{E}_{Rect}^{(N)} = \mathcal{E}_{Rect}^{(D)} - (1 + (a/b)^2)\frac{\zeta(3)}{16\pi a^2}$$

3/ E&M