EXTERIOR EULER SUMMABILITY

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Abstract. We define and study a summability procedure that is similar to Euler summability but applied in the exterior of a disc, not in the interior. We show that the method is well defined and that it actually has many interesting properties.

We use these ideas to give a good definition of the support of an analytic functional. We study some problems of analytic continuation and also study Mittag-Leffler developments using this exterior summability method.

1. Introduction

In a recent article, Amore [1] introduced a method for the convergence acceleration of series, illustrating the procedure with several very interesting examples. This method is similar, and actually related to the method introduced by Euler many years ago [11, Chp. VIII]. Basically, while Euler's method works in the *interior* of a disc, Amore's procedure works in the *exterior* of a disc.

The purpose of this article is to study this exterior procedure not as a convergence acceleration method but rather as a *summability* method. The study of the Euler interior procedure as a summability method is due initially to Knopp [13, 14], and it is clearly explained in [11]; examples of Euler's method for the convergence acceleration of series can be found in [4]. As we show, the exterior Euler summability is not only different from the usual Euler summability, but it actually provides many new interesting and useful results, particularly in the study of analytic continuations and in the definition of the support of analytic functionals.

The plan of the article is the following. We define two notions of exterior Euler summability in Section 2, one that applies to power series and one that applies to numerical series, and prove that both definitions are equivalent. In Section 3 we prove that if one starts with a series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ which converges for $|z| > R$ for some $R < \infty$, then there exists a compact convex set K_{cv} such that the series is

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exterior Euler summable for $z \notin K_{\mathrm{cv}}$ but is never exterior Euler summable if $z \in K_{\mathrm{cv}}$ is not an extreme point of this set; both summability and non summability are possible at the extreme points. We interpret these results in terms of the analytic continuation of the function $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ defined in $|z| > R$, showing that it has an analytic continuation f_{cv} to $\overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$, and that K_{cv} is the smallest compact convex subset of $\mathbb{C}$ for which such a continuation exists.

The exterior Euler summability is based upon a series transformation in terms of a parameter λ. In Section 4 we prove that the sum of the transformed series, if it exists, is independent of this parameter; the uniqueness is rather clear if $z \notin K_{\mathrm{cv}}$, but it involves the study of the boundary behavior of $f_{\mathrm{cv}}(\omega)$ as $\omega \to z$ if z is an extreme point of K_{cv}.

Section 5 gives an account of the relationship between the exterior Euler summability and some results about analytic functionals, hyperfunctions, and distributions. The use of the Cauchy representation [8] allows one to relate an analytic function to an analytic functional, and the series transformation required in the Euler exterior summability becomes a change of variables in the corresponding analytic functional.

We present several illustrations in Section 6. In particular we show that the geometric series $\sum_{n=0}^{\infty} \omega^n$ is exterior Euler summable, to the sum $(1-\omega)^{-1}$, for *all* complex numbers $\omega \neq 1$, and this gives an idea of the power of the method; the standard interior Euler method gives a much smaller region of summability for the geometric series [11]. In Section 7 we explain how not only convergent, but actually Abel summable series can be transformed by the exterior Euler procedure, and illustrate the ideas with the series $\sum_{n=1}^{\infty} (-1)^n n^{-s}$, which is Abel summable for any complex number s.

The exterior Euler expansion of functions given by Mittag-Leffler developments is studied in Section 8, where we show that the region of summability is rather large, and where we give several examples of this "double series" manipulations.

2. Definition of Exterior Euler Summability

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Suppose that the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ converges for $|z| > R$ for some $R < \infty$. Let $\xi \in \mathbb{C}$. We say that the series $\sum_{n=0}^{\infty} a_n/\xi^{n+1}$ is exterior Euler summable to $S = S(\xi)$, and write

$$\sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} = S \qquad (\mathrm{Ex}) \ , \tag{2.1}$$

if there exists $\lambda \in \mathbb{C}$ such that

$$\sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{(\xi+\lambda)^{n+1}} = S\,, \tag{2.2}$$

where

$$a_{n,\lambda} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} a_j\,. \tag{2.3}$$

When $\lambda = 0$ it reduces to ordinary convergence.

One may think that it is possible, in principle, that the sum value S given by (2.2) depends on λ, $S = S_\lambda$. However, we shall show in Section 4 that if the corresponding series converge at λ_1 and at λ_2, then $S_{\lambda_1} = S_{\lambda_2}$.

Our definition makes it clear that the exterior Euler summability is to be applied to a *power series* in ξ^{-1}. One may give a corresponding definition for *numerical series.* We say that the series $\sum_{n=0}^{\infty} a_n$ is (Ex') summable to S if $\sum_{n=0}^{\infty} a_n/\xi^{n+1} = S$ (Ex) ,when $\xi = 1$. Fortunately the two summability methods are equivalent.

Lemma 2.1. *Let $\{a_n\}_{n=0}^{\infty}$ be a numerical sequence and let $\xi \in \mathbb{C}$. Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} = S \qquad (\mathit{Ex'})\,, \tag{2.4}$$

if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\xi^{n+1}} = S \qquad (\mathit{Ex})\,. \tag{2.5}$$

Proof. Let $A_n(\xi) = a_n/\xi^{n+1}$. Then $\sum_{n=0}^{\infty} a_n/\xi^{n+1} = S$ (Ex') , if and only if $\sum_{n=0}^{\infty} A_n(\xi)/\omega^{n+1} = S$ (Ex) , for $\omega = 1$, and this, in turn, is equivalent to the existence of $\lambda \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} A_{n,\lambda}(\xi)/(1+\lambda)^{n+1} =$

S, where

$$\begin{aligned} A_{n,\lambda}(\xi) &= \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} A_j(\xi) \\ &= \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \frac{a_j}{\xi^{j+1}} \\ &= \frac{1}{\xi^{n+1}} \sum_{j=0}^{n} \binom{n}{j} (\lambda\xi)^{n-j} a_j \\ &= \frac{1}{\xi^{n+1}} a_{n,\lambda\xi} \,. \end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} A_{n,\lambda}(\xi) / (1+\lambda)^{n+1} = S$, is equivalent to the convergence of $\sum_{n=0}^{\infty} a_{n,\lambda\xi} / (\xi + \lambda\xi)^{n+1}$ to S, and this means exactly that $\sum_{n=0}^{\infty} a_n/\xi^{n+1} = S$ (Ex). □

3. The Region of Summability

We now study the set of points where a power series of the type $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable. Naturally, there is a disc $D = \mathbb{D}(0, r_0)$ such that the series converges for $z \notin \overline{D}$ and diverges for $z \in D$. In the case of (Ex) summability we shall show that there exists a compact convex set K such that $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable if $z \notin K$ while it is not exterior Euler summable if $z \in \operatorname{Int} K$. As in the convergence case, both summability and non-summability can occur if $z \in \partial K$. In fact we shall show that

$$K = \bigcap_{\lambda \in \mathbb{C}} \overline{D}_\lambda \,, \tag{3.1}$$

where $D_\lambda = \mathbb{D}(-\lambda, r_\lambda)$ is the disc centered at $-\lambda$ outside of where the series $\sum_{n=0}^{\infty} a_{n,\lambda} / (z+\lambda)^{n+1}$ is convergent.

We shall also establish that the analytic function

$$f_0(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \,, \qquad |z| > r_0 \,, \tag{3.2}$$

has a *unique* analytic continuation to the region $\overline{\mathbb{C}} \setminus K$.

Observe that if the analytic function g_0, defined in a region Ω_0, admits two analytic continuations, g_1 and g_2, defined in corresponding regions Ω_1 and Ω_2, then in general it is not true that $g_1(z) = g_2(z)$ if $z \in \Omega_1 \cap \Omega_2$. (Consider for example $g_0(z)$ the branch of $\ln z$ defined in $|z - i| < 1$, with $g_0(i) = \pi i/2$, while g_1 is the analytic continuation to $\mathbb{C}\setminus[0,\infty)$ and g_2 is the analytic continuation to $\mathbb{C}\setminus(-\infty,0]$; then $g_1(-i)=3\pi i/2$ but $g_2(-i)=-\pi i/2$.)

However we have:

Lemma 3.1. *Let $f_0(z)=\sum_{n=0}^{\infty}a_n/z^{n+1}$ for $z\in\Omega_0=\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(0,r)$, the series being convergent in Ω_0. Suppose that $f_0\neq 0$. Let K_1 and K_2 be two compact convex subsets of $\mathbb{C}$. Suppose f_0 admits analytic continuations f_j, $j=1,2$, defined in $\Omega_j=\overline{\mathbb{C}}\setminus K_j$. Then*

$$K_1\cap K_2\neq\emptyset\,, \tag{3.3}$$

$$f_1(z)=f_2(z)\quad \textit{for}\quad z\in\Omega_1\cap\Omega_2\,, \tag{3.4}$$

and f_0 admits an analytic continuation to $\Omega_1\cup\Omega_2$, which is also an analytic continuation of f_1 and f_2.

Proof. Let us first show that $f_1(z)=f_2(z)$ for $z\in\Omega_1\cap\Omega_2$. Since $f_0=f_1=f_2$ in Ω_0, it follows that $f_1=f_2$ in the component of $\Omega_1\cap\Omega_2$ that contains Ω_0; hence (3.3) will follow if we show that $\Omega_1\cap\Omega_2$ is connected. But if $z\in\mathbb{C}\setminus(K_1\cup K_2)$ there is a ray from z to ∞ that does not meet $K_1\cup K_2$. Indeed, let A_j be the set of complex numbers η, $|\eta|=1$, such that the ray $z+t\eta$, $t>0$, meets K_j. If $z\notin K_j$ then A_j is an arc of $|\eta|=1$ of length $|A_j|<\pi$ since K_j is convex. Hence $|A_1\cup A_2|<2\pi$, and thus there exists $\eta\in\mathbb{C}$, $|\eta|=1$, such that $\eta\notin A_1\cup A_2$, that is, such that the ray $z+t\eta$, $t>0$ does not meet $K_1\cup K_2$. Since any point of $\Omega_1\cap\Omega_2=\overline{\mathbb{C}}\setminus(K_1\cup K_2)$ can be joined to ∞ by a ray contained in $\Omega_1\cap\Omega_2$ it follows that $\Omega_1\cap\Omega_2$ is arcwise connected and thus connected.

If we now define $f(z)=f_j(z)$ if $z\in\Omega_j$, $j=1,2$, for $z\in\Omega_1\cup\Omega_2$, it follows that f is well defined, analytic, and an analytic continuation of f_0, f_1, and f_2.

Finally, if $K_1\cap K_2=\emptyset$ it would follow that $\Omega_1\cup\Omega_2=\overline{\mathbb{C}}$, and hence f_0 would admit an analytic continuation f to the whole Riemann sphere. But by Liouville theorem f would be constant, and since $f_0(\infty)=0$, then $f=0$, a contradiction. ☐

Using induction we then have,

Lemma 3.2. *Let $f_0(z)=\sum_{n=0}^{\infty}a_n/z^{n+1}$ for $z\in\Omega_0=\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}(0,r)$, the series being convergent in Ω_0. Suppose that $f_0\neq 0$. Let $K_1,\ldots,K_m$ be compact convex subsets of $\mathbb{C}$. Suppose f_0 admits analytic continuations f_j, $j=1,\ldots,m$, defined in $\Omega_j=\overline{\mathbb{C}}\setminus K_j$. Then*

$$\bigcap_{j=1}^{m}K_j\neq\emptyset\,, \tag{3.5}$$

$$f_j(z) = f_k(z) \quad \textit{for} \quad z \in \Omega_j \cap \Omega_k\,, \tag{3.6}$$

and $f_0, f_1, \ldots, f_m$ *admit an analytic continuation to* $\bigcup_{j=1}^m \Omega_j$.

Let us now consider the family $\mathcal{F}$ of analytic continuations (f_Ω, Ω) of (f_0, Ω_0) such that $\Omega = \overline{\mathbb{C}} \setminus L$ where L is a compact convex set. Then the family of these sets L has the finite intersection property, because of (3.5). Hence we can define

$$K_{\mathrm{cv}} = \bigcap_{(f,\Omega)\in\mathcal{F}} \overline{\mathbb{C}} \setminus \Omega\,, \tag{3.7}$$

and the analytic extension f_{cv} defined in $\overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$. Observe that K_{cv} is the *smallest* compact convex set such that f_0 admits an analytic continuation defined in its complement.

We shall show that the power series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable to $f_{\mathrm{cv}}(z)$ for $z \in \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$, but the series is never exterior Euler summable if $z \in \operatorname{Int} K_{\mathrm{cv}}$. We start with the following lemma.

Lemma 3.3. *Let* $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ *for* $z \in \Omega_0 = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(0,r)$, *the series being convergent in* Ω_0. *Suppose that* $f_0 \neq 0$. *Then*

$$K_{\mathrm{cv}} = \bigcap_{\lambda\in\mathbb{C}} \overline{D}_\lambda\,, \tag{3.8}$$

where $D_\lambda = \mathbb{D}(-\lambda, r_\lambda)$ *is the disc such that* $\sum_{n=0}^{\infty} a_{n,\lambda}/(\xi+\lambda)^{n+1}$ *converges if* $\xi \notin \overline{D}_\lambda$ *and diverges if* $\xi \in D_\lambda$.

Proof. Let $K_{\mathrm{e}} = \bigcap_{\lambda\in\mathbb{C}} \overline{D}_\lambda$. If $\mathcal{F}_1$ is a subfamily of the family $\mathcal{F}$ of analytic continuations (f_Ω, Ω) of (f_0, Ω_0) such that $\Omega = \overline{\mathbb{C}} \setminus L$ where L is a compact convex set, then

$$K_{\mathrm{cv}} \subseteq \bigcap_{(f,\Omega)\in\mathcal{F}_1} \overline{\mathbb{C}} \setminus \Omega\,,$$

and so $K_{\mathrm{cv}} \subseteq K_{\mathrm{e}}$.

To prove the reverse inclusion, let us observe that if $K_{\mathrm{cv}} \subset D$, where $D = \mathbb{D}(-\lambda, s)$ is a disc with center at $-\lambda$, then $s > r_\lambda$ and actually $\sum_{n=0}^{\infty} a_{n,\lambda}/(\xi+\lambda)^{n+1}$ converges if $\xi \in \overline{\mathbb{C}} \setminus D$. If we now use the fact that for any compact convex set K we have $K = \bigcap_{\substack{K\subset D\\ D \text{ open disc}}} D$, we obtain

$$K_{\mathrm{e}} = \bigcap_{\lambda\in\mathbb{C}} \overline{D}_\lambda \subseteq \bigcap_{\substack{K_{\mathrm{cv}}\subset D\\ D \text{ open disc}}} D = K_{\mathrm{cv}}\,, \tag{3.9}$$

as required. □

In the proof of the previous lemma we used that any compact convex subset of $\mathbb{C}$ is the intersection of all the open discs that contain it. We now give a proof of this simple fact along with other related results.

Lemma 3.4. *Let K be a compact convex subset of $\mathbb{C}$. Then,*

$$K = \bigcap_{\substack{K \subset D \\ D \text{ open disc}}} D \,, \tag{3.10}$$

and also

$$K = \bigcap_{\substack{K \subset D \\ D \text{ open disc}}} \overline{D} \,. \tag{3.11}$$

On the other hand

$$\bigcap_{\substack{K \subset \overline{D} \\ D \text{ open disc}}} D = \{ z \in K : z \text{ is not an extreme point of } K \} \,. \tag{3.12}$$

Proof. To prove (3.10) it is enough to prove that if $z \notin K$ then there exists an open disc D with $K \subset D$ and $z \notin D$. Since there exists an open half plane H with the property that $K \subset H$ and $z \notin H$, we can assume that $K \subset \{\omega \in \mathbb{C} : \Re e\, \omega < 0\}$ and $z > 0$. Since K is compact, there exists $M > 0$ such that $K \subset \{\omega \in \mathbb{C} : -M \leq \Re e\, \omega < 0,\ |\Im m\, \omega| \leq M\}$. Choose $\lambda > 0$ such that

$$\lambda > \max \left\{ M, \frac{M^2 - z^2}{2z} \right\} , \tag{3.13}$$

and r such that

$$\sqrt{\lambda^2 + M^2} < r < \lambda + z \,. \tag{3.14}$$

Then $K \subset \mathbb{D}\left(-\lambda, r\right)$ and $z \notin \mathbb{D}\left(-\lambda, r\right)$.

Next we observe that clearly $K \subset \bigcap_{K \subset D,\ D \text{ open disc}} \overline{D}$, while if $K \subset D$, where $D = \mathbb{D}\left(-\lambda, s\right)$, then $r = \max_{z \in K} |z + \lambda| < s$, and so if $r < t < s$, then $K \subset D_1 \subset \overline{D}_1 \subset D$, if $D_1 = \mathbb{D}\left(-\lambda, t\right)$. Hence $\bigcap_{K \subset D_1,\ D_1 \text{ open disc}} \overline{D}_1 \subseteq \bigcap_{K \subset D,\ D \text{ open disc}} D = K$. This gives (3.11).

Finally we establish (3.12). Observe first that $L = \bigcap_{K \subset \overline{D},\ D \text{ open disc}} D$ is a subset of K. If $z \in K$ is not an extreme point, then there exists $\omega_1, \omega_2 \in K$ such that z is in the open segment from ω_1 to ω_2. If $K \subset \overline{D}$, D an open disc, then since $\omega_1, \omega_2 \in \overline{D}$ it follows that $z \in D$; thus $z \in L$. On the other hand, if z is an extreme point of K, then there exists an open disc D with $K \subset \overline{D}$ and with $\partial D \cap K = \{z\}$, and this yields that $z \notin L$. $\square$

Returning to exterior Euler summability, we can give the following result.

Theorem 3.5. *Let* $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ *for* $z \in \Omega_0 = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(0,r)$, *the series being convergent in* Ω_0. *Suppose that* $f_0 \neq 0$. *Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = f_{\mathrm{cv}}(z) \qquad (Ex) , \tag{3.15}$$

if $z \in \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$. *The series* $\sum_{n=0}^{\infty} a_n/z^{n+1}$ *is not exterior Euler summable if* $z \in K_{\mathrm{cv}}$ *is not an extreme point of* K_{cv}.

Proof. Notice that the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable if $z \notin K_{\mathrm{e}} = \bigcap_{\lambda \in \mathbb{C}} \overline{D}_\lambda$, thus the Lemma 3.3 yields (3.15).

Suppose now that $z \in K_{\mathrm{cv}}$ and $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is exterior Euler summable. Then there exists $\lambda \in \mathbb{C}$ such that $z \in \overline{\mathbb{C}} \setminus D_\lambda$. But $K_{\mathrm{cv}} \subset \overline{D}_\lambda$, so $z \notin \bigcap_{K_{\mathrm{cv}} \subset \overline{D},\ D \text{ open disc}} D$, and from (3.12), we obtain that z must be an extreme point of K_{cv}. □

Let us remark that the function f_{cv} may or may not have an analytic extension across the flat portions of ∂K_{cv}, but the extreme points of K_{cv} are natural boundary points for the analytic continuation. In particular, if ∂K_{cv} is strictly convex, so that it does not have any flat sections, then ∂K_{cv} is a natural boundary for the analytic continuation of f_{cv}. When ∂K_{cv} is a natural boundary for the analytic continuation of f_{cv} then $\left(f_{\mathrm{cv}}, \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}\right)$ is the maximal analytic continuation of f_0, not only among the ones defined in the complement of a convex set, but among *all* analytic continuations. Interestingly, the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ could be exterior Euler summable at the extreme points of ∂K_{cv} but it never is in the flat sections of ∂K_{cv}.

4. Uniqueness of the Sum

Implicit in our definition of exterior Euler summability is the fact that for a fixed $z \in \mathbb{C}$ the sum of the series $\sum_{n=0}^{\infty} a_{n,\lambda}/\left(z+\lambda\right)^{n+1}$, if convergent, is independent of λ. We now show that this is the case.

If $z \in \overline{\mathbb{C}} \setminus K_{\mathrm{cv}}$, then (3.15) shows that $\sum_{n=0}^{\infty} a_{n,\lambda}/\left(z+\lambda\right)^{n+1} = f_{\mathrm{cv}}(z)$ whenever the series is convergent, and thus the sum of the series is independent of λ. When $z \in K_{\mathrm{cv}}$, however, a proof of the uniqueness of the sum value is required. Let us start with some preliminary results.

Lemma 4.1. *Let $z \in K_{\mathrm{cv}}$. If the series $\sum_{n=0}^{\infty} a_{n,\lambda}/\left(z+\lambda\right)^{n+1}$ converges, then*

$$\lim_{t \to 0^{+}} f_{\mathrm{cv}}\left(z+t\omega\right) = \sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{\left(z+\lambda\right)^{n+1}}\,, \tag{4.1}$$

whenever

$$\Re e\left(\frac{\omega}{z+\lambda}\right) > 0\,, \tag{4.2}$$

uniformly on compacts of this open half-plane $\mathbb{H}_{\lambda}$.

If $\mu \in \mathbb{C}$ let

$$h_{\mu}\left(\omega\right) = \sum_{n=2}^{\infty} \frac{a_{n,\mu}}{\left(n-1\right)n\left(z+\mu\right)^{n-1}}\,, \tag{4.3}$$

for $\omega \in \mathbb{C}\setminus\overline{D}_{\mu}$. Then $h_{\lambda}\left(\omega\right)$ admits a continuous extension to $\mathbb{C}\setminus D_{\lambda}$, while $\left(\omega-z\right)^{2}h_{-z}\left(\omega\right)$ admits a continuous extension to $\mathbb{C}\setminus D_{\lambda}$ that vanishes at $\omega = z$.

Proof. The limit formula (4.1) when (4.2) is satisfied follows easily from the Abel limit theorem. For the second part, observe that if $\sum_{n=0}^{\infty} a_{n,\lambda}/\left(z+\lambda\right)^{n+1}$ converges, then the series defining $h_{\lambda}\left(\omega\right)$ is absolutely convergent if $\left|\omega+\lambda\right| \geq \left|z+\lambda\right|$, and this yields the continuity of $h_{\lambda}\left(\omega\right)$ in $\mathbb{C}\setminus D_{\lambda}$. The result about $\left(\omega-z\right)^{2}h_{-z}\left(\omega\right)$ follows by writing this function in terms of $h_{\lambda}\left(\omega\right)$, observing that

$$h_{\mu_1}\left(\omega\right) - h_{\mu_2}\left(\omega\right) = \left(a_0\omega - a_1\right)\ln\left(\frac{\omega+\mu_1}{\omega+\mu_2}\right) + a_0\left(\mu_2-\mu_1\right)\,, \tag{4.4}$$

for $\omega \in \mathbb{C}\setminus\left(\overline{D}_{\mu_1}\cup\overline{D}_{\mu_2}\right)$. □

Observe that, in particular,

$$\lim_{t \to 0^{+}} f_{\mathrm{cv}}\left(z+t\left(z+\lambda\right)\right) = \sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{\left(z+\lambda\right)^{n+1}}\,, \tag{4.5}$$

if the series converges.

Theorem 4.2. *Let $f_0\left(z\right) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ for $z \in \Omega_0 = \overline{\mathbb{C}}\setminus\overline{\mathbb{D}}\left(0,r\right)$, the series being convergent in Ω_0. Suppose that $f_0 \neq 0$. Let $z \in \mathbb{C}$. Suppose the series*

$$S_{\lambda} = \sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{\left(z+\lambda\right)^{n+1}}\,, \tag{4.6}$$

converges for two complex numbers λ_1 *and* λ_2*. Then*

$$S_{\lambda_1} = S_{\lambda_2}\,. \tag{4.7}$$

Proof. It remains to give the proof when $z \in K_{\mathrm{cv}}$. Observe first that unless z belongs to the segment from $-\lambda_1$ to $-\lambda_2$, then the two half-planes $\mathbb{H}_{\lambda_1}$ and $\mathbb{H}_{\lambda_2}$ cannot be disjoint, and thus if $\omega \in \mathbb{H}_{\lambda_1} \cap \mathbb{H}_{\lambda_2}$, the Lemma 4.1 yields that $\lim_{t\to 0^+} f_{\mathrm{cv}}(z + t\omega)$ should be equal to both S_{λ_1} and S_{λ_2}, and (4.7) follows.

If z belongs to the segment from $-\lambda_1$ to $-\lambda_2$, then $K_{\mathrm{cv}} = \{z\}$, and thus $f_{\mathrm{cv}}(\omega) = g\left((\omega - z)^{-1}\right)$ for some entire function g with $g(0) = 0$. Using the Lemma 4.1, the function $(\omega - z)^2 h_{-z}(\omega)$ is continuous in $\mathbb{C} \setminus D_{\lambda_1}$, and continuous in $\mathbb{C} \setminus D_{\lambda_2}$. But $(\mathbb{C} \setminus D_{\lambda_1}) \cup (\mathbb{C} \setminus D_{\lambda_2}) = \mathbb{C}$, and thus $(\omega - z)^2 h_{-z}(\omega)$ is continuous in all $\mathbb{C}$. Moreover, h_{-z} is analytic at ∞, with $h_{-z}(\infty) = 0$, and therefore $h_{-z}(z + 1/\xi) = A + B\xi$ for some constants A and B. It follows that g is a polynomial of degree 3 at the most. However, if g is a polynomial, then $\lim_{t\to 0^+} f_{\mathrm{cv}}(z + t\omega) = \infty$ for any $\omega \neq 0$, which implies that $S_{\lambda_1} = S_{\lambda_2} = \infty$, and thus the series $\sum_{n=0}^{\infty} a_{n,\lambda}/(z + \lambda)^{n+1}$ diverges for both $\lambda = \lambda_1$ and $\lambda = \lambda_2$. $\square$

Our analysis gives the behavior of $f_{\mathrm{cv}}(\omega)$ as ω approaches an extreme point of K_{cv} where the series is exterior Euler summable.

Theorem 4.3. *Let* $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ *for* $z \in \Omega_0 = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, r)$*, the series being convergent in* Ω_0*. Suppose that* $f_0 \neq 0$*. Let* $z \in K_{\mathrm{cv}}$ *be a point where*

$$\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = S \qquad (Ex)\,, \tag{4.8}$$

exists. Then there exists an open arc I *of* $|\xi| = 1$ *with* $|I| \geq \pi$ *such that*

$$\lim_{t\to 0^+} f_{\mathrm{cv}}\left(z + te^{i\theta}\right) = S\,, \tag{4.9}$$

if $e^{i\theta} \in I$*, uniformly over compacts of* I*. There may be rays* $z + te^{i\theta}$*,* $t > 0$*, of* $\mathbb{C} \setminus K_{\mathrm{cv}}$ *for which the limit of* $f_{\mathrm{cv}}\left(z + te^{i\theta}\right)$ *as* $t \to 0^+$ *does not exist.*

Proof. Indeed, we just need to take I as the set of numbers ξ with $|\xi| = 1$ that belong to some half-plane $\mathbb{H}_\lambda$ given be (4.2) for which the series (4.1) converges.

If we take

$$f_{\mathrm{cv}}(\omega) = \omega^2 e^{-1/\omega} - \omega^2 + \omega - \frac{1}{2}\,, \tag{4.10}$$

then

$$f_{\mathrm{cv}}(\omega)=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(n+3)!\omega^{n+1}} \qquad (\mathrm{Ex}) , \tag{4.11}$$

for $\omega \neq 0$, so that $K_{\mathrm{cv}}=\{0\}$. The series is exterior Euler summable at $\omega=0$ to $S=-1/2$. Here $f_{\mathrm{cv}}(t\omega)\to S$ as $t\to 0^{+}$ if $\Re e\,\omega\geq 0$, but the limit does not exist if $\Re e\,\omega<0$. □

Observe that while f_{cv} is analytic, and thus continuous in $\overline{\mathbb{C}}\setminus K_{\mathrm{cv}}$, its extension to the set of points where the series is exterior Euler summable, a subset of $(\overline{\mathbb{C}}\setminus K_{\mathrm{cv}})\cup\partial K_{\mathrm{cv}}$, will not be continuous, in general.

5. Analytic Functionals and Exterior Summability

There is a close connection between the exterior Euler summability and some results about analytic functionals and hyperfunctions [15, 16, 17], as we now explain.

Let U be an open set in $\mathbb{C}$. We denote by $\mathfrak{O}(U)$ the space of analytic functions defined on U. The topology of $\mathfrak{O}(U)$ is that of uniform convergence on compact subsets of U, i.e., the topology generated by the family of seminorms $\|\varphi\|_{K}=\max\{|\varphi(z)|: z\in K\}$, for K a compact subset of U and $\varphi\in\mathfrak{O}(U)$. Since we can find a sequence of compact subsets of U, $\{K_n\}_{n=1}^{\infty}$, with $K_n\subset \operatorname{int}(K_{n+1})$, $\bigcup_{n=1}^{\infty}K_n=U$, it follows that $\mathfrak{O}(U)$ is a Fréchet space, actually a strict projective limit of Banach spaces.

A subset S of a topological space X is called locally closed if each $x\in S$ has a neighborhood in X, V_x, such that $S\cap V_x$ is closed in V_x. It can be shown that S is locally closed in X if and only if there exist an open set U and a closed set F such that $S=U\cap F$. If S is locally closed in X, we say that U is an open neighborhood of S if U is open in X and S is closed in U. We denote the set of open neighborhoods of S as $\mathsf{N}(S)$.

If S is locally closed in $\mathbb{C}$ then $\mathfrak{O}(S)$ is the space of germs of analytic functions defined on S. That is, a function φ defined on S belongs to $\mathfrak{O}(S)$ if and only if there exists $U\in\mathsf{N}(S)$ and an analytic function $\widetilde{\varphi}\in\mathfrak{O}(U)$ such that $\pi_S^U(\widetilde{\varphi})=\varphi$, where π_S^U is the restriction operator from U to S. The system of topological vector spaces $\{\mathfrak{O}(U)\}_{U\in\mathsf{N}(S)}$ with operators $\pi_V^U:\mathfrak{O}(U)\longrightarrow\mathfrak{O}(V)$ for $U\supseteq V$ is actually a directed system and thus we can give $\mathfrak{O}(S)$ the inductive limit topology. When K is compact, then $\mathfrak{O}(K)$ is a strict limit of Banach spaces. If $S\subseteq\mathbb{R}$ is open then $\mathfrak{O}(S)$ is the space of real analytic functions on S, while if

$S \subseteq \mathbb{R}$ is locally closed then $\mathfrak{O}(S)$ is the space of germs of real analytic functions on S.

If $S \subseteq \mathbb{C}$ is locally closed, then the dual space $\mathfrak{O}'(S)$ is called the space of *analytic functionals* on S. When $K \subseteq \mathbb{R}$ is compact then $\mathfrak{O}'(K)$ is actually isomorphic to the space $\mathfrak{B}(K)$ of hyperfunctions defined on K, although hyperfunctions are usually constructed by using a different approach [15]. Observe that if $K \subseteq \mathbb{R}$ then the space of distributions $T \in \mathcal{D}'(\mathbb{R})$ whose support is contained in K, the space $\mathcal{E}'[K]$, is a subspace of $\mathfrak{B}(K)$.

If K is a compact subset of $\mathbb{C}$, and $T \in \mathfrak{O}'(K)$ then its Cauchy or analytic representation, denoted as $f(z) = \mathcal{C}\{T(\omega); z\}$, is the analytic function $f \in \mathfrak{O}\left(\overline{\mathbb{C}} \setminus K\right)$ given by

$$f(z) = \mathcal{C}\{T(\omega); z\} = \frac{1}{2\pi i}\left\langle T(\omega), \frac{1}{\omega - z}\right\rangle . \tag{5.1}$$

Notice that the analytic representation satisfies

$$\lim_{z\to\infty} f(z) = 0 . \tag{5.2}$$

According to a theorem of Silva [15], the operator $\mathcal{C}$ is an isomorphism of the space $\mathfrak{O}'(K)$ onto the subspace $\mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$ of $\mathfrak{O}\left(\overline{\mathbb{C}} \setminus K\right)$ formed by those analytic functions that satisfy (5.2). When $K \subseteq \mathbb{R}$ then the operator $\mathcal{C}$ becomes an isomorphism of the space of hyperfunctions $\mathfrak{B}(K)$ onto $\mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$.

The inverse operator $\mathcal{C}^{-1}$ is given as follows. Let $\varphi \in \mathfrak{O}(K)$, and let $\widetilde{\varphi} \in \mathfrak{O}(U)$ be an analytic extension to some region $U \in \mathsf{N}(K)$; let C be a closed curve in U such that the index of any point of K with respect to C is one. Then if $f \in \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$ we define $T = \mathcal{C}^{-1}\{f\} \in \mathfrak{O}'(K)$ by specifying its action on φ as

$$\langle T(\omega), \varphi(\omega)\rangle = -\oint_C f(\xi)\, \widetilde{\varphi}(\xi)\, \mathrm{d}\xi . \tag{5.3}$$

Clearly $T = \mathcal{C}^{-1}\{f\}$ is defined if $f \in \mathfrak{O}\left(\overline{\mathbb{C}} \setminus K\right)$, but in this space $\mathcal{C}^{-1}$ has a non trivial kernel, namely, the constant functions.

If K_1 and K_2 are compact subsets of $\mathbb{C}$ with $K_1 \subset K_2$ and K_2 has no holes, that is, $\overline{\mathbb{C}} \setminus K_2$ is connected, then any functional $T \in \mathfrak{O}'(K_1)$ can be considered as an analytic functional of the space $\mathfrak{O}'(K_2)$, so that we have a canonical injection $\mathfrak{O}'(K_1) \hookrightarrow \mathfrak{O}'(K_2)$. This injection corresponds to the injection $\mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K_1\right) \hookrightarrow \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K_2\right)$ provided by the restriction to a smaller region. In general not all analytic functionals $T \in \mathfrak{O}'(K_2)$ are in the image of $\mathfrak{O}'(K_1)$, that is, in general they do not admit an "extension" to $\mathfrak{O}'(K_1)$. An extension exists precisely when the Cauchy representation $f = \mathcal{C}\{T\} \in \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K_2\right)$ admits an analytic continuation to $\overline{\mathbb{C}} \setminus K_1$.

If $T \in \mathfrak{O}'(K)$, then the power series expansion of its Cauchy representation at infinity takes the form

$$\mathcal{C}\{T(\omega);z\} = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\mu_n(T)}{z^{n+1}}, \qquad |z| > \rho, \tag{5.4}$$

where

$$\mu_n(T) = \langle T(\omega), \omega^n\rangle, \tag{5.5}$$

are the moments of T and where $\rho = \max\{|z| : z \in K\}$. Observe that $\mathcal{C}\{T(\omega);z\}$ is defined if $z \in \overline{\mathbb{C}} \setminus K$, but the series in (5.4) will be divergent if $|z| < \rho$, and could be divergent if $|z| = \rho$.

Our results about exterior Euler summability yield several corresponding results on analytic functionals.

Theorem 5.1. *Let K be a compact convex subset of $\mathbb{C}$ and let $T \in \mathfrak{O}'(K)$ with Cauchy representation $f = \mathcal{C}\{T\} \in \mathfrak{O}_0\left(\overline{\mathbb{C}} \setminus K\right)$. Then*

$$\sum_{n=0}^{\infty}\frac{\mu_n(T)}{z^{n+1}} = -2\pi i f(z) \qquad (Ex), \tag{5.6}$$

for all $z \in \mathbb{C} \setminus K$.

We also obtain the following result.

Theorem 5.2. *Let $f_0(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ for $z \in \Omega_0 = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(0,r)$, the series being convergent in Ω_0. Suppose that $f_0 \neq 0$. Let $T_0 \in \mathfrak{O}'\left(\overline{\mathbb{D}}(0,r)\right)$ be the analytic functional $T_0 = \mathcal{C}^{-1}\{f_0\}$. Then there exists a smallest compact convex subset $K \subset \overline{\mathbb{D}}(0,r)$ such that T_0 admits an extension to $T \in \mathfrak{O}'(K)$. This smallest compact convex set is actually K_{cv} and the analytic representation of T is f_{cv}.*

Using this result we obtain that one can define the notion of the *compact convex support* of an analytic functional. Naturally the notion of the "support" of an analytic functional is not well defined.

The solution of the moment problem in the space of distributions $\mathcal{E}'[I]$, where I is an interval of the form $[-a,a]$, was given in [7] (see also [9, Thm. 7.3.1]). Here we can give the solution of moment problems in $\mathfrak{O}'(K)$ if K is a compact convex subset of $\mathbb{C}$.

Theorem 5.3. *Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and let K be a compact convex subset of $\mathbb{C}$. Then the moment problem*

$$\langle T(\omega), \omega^n\rangle = \mu_n, \qquad n \in \mathbb{N}, \tag{5.7}$$

has a solution $T \in \mathfrak{O}'(K)$ if and only if the series $\sum_{n=0}^{\infty} \mu_n / z^{n+1}$ is exterior Euler summable for all $z \in \mathbb{C} \setminus K$, in fact, $-2\pi i T$ is then the inverse Cauchy representation of the analytic function given by the Euler exterior sum of this series. If there is a solution, it is unique.

6. Special Cases

We shall now give several examples of the exterior Euler summability.

Example 1. Let $\omega \in \mathbb{C}$ be *fixed* and consider the Dirac delta function at ω, the analytic functional $\delta_\omega \in \mathfrak{O}'(\{\omega\})$, given by

$$\langle \delta_\omega(z), \varphi(\omega) \rangle = \varphi(\omega) , \tag{6.1}$$

for $\varphi \in \mathfrak{O}(\{\omega\})$. In this case the moments are given by $\mu_n = \omega^n$, $n \in \mathbb{N}$, while the Cauchy representation is $f(z) = (2\pi i)^{-1} (\omega - z)^{-1}$, so that we obtain,

$$\sum_{n=0}^{\infty} \frac{\omega^n}{z^{n+1}} = \frac{1}{z - \omega} \qquad (\mathrm{Ex}) , \tag{6.2}$$

for all $z \neq \omega$.

The power of a summation method is many times measured [11] by the set of points ω where the geometric series $\sum_{n=0}^{\infty} \omega^n$ is summable, to $(1-\omega)^{-1}$, of course. For instance for convergence this set is the open disc $|\omega| < 1$, while for Cesàro summability it is the set $|\omega| \leq 1$, $\omega \neq 1$. We now study this question for (Ex') summability.

Example 2. The geometric series $\sum_{n=0}^{\infty} \omega^n$ is (Ex') summable if and only if the series $\sum_{n=0}^{\infty} \omega^n / z^{n+1}$ is (Ex) summable at $z = 1$; the previous example shows that this is the case precisely when $1 = z \neq \omega$. Therefore,

$$\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega} \qquad (\mathrm{Ex'}) , \qquad \text{for all } \omega \neq 1 . \tag{6.3}$$

It is easy to see that one can take the derivative of an exterior Euler summation formula in the region $\mathbb{C} \setminus K_{\mathrm{cv}}$. This yields the following formulas.

Example 3. If $k \in \mathbb{N}$, $k \geq 1$, differentiation of (6.2) yields the formula

$$\sum_{n=k-1}^{\infty} \binom{n}{k-1} \frac{\omega^{n+1-k}}{z^{n+1}} = \frac{1}{(z-\omega)^k} \qquad (\mathrm{Ex}) , \qquad \text{for } z \neq \omega . \tag{6.4}$$

Hence, we also obtain

$$(6.5)\quad \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}\omega^{n}=\frac{1}{(1-\omega)^{k}}\qquad (\text{Ex'})\ ,\qquad \text{for all } \omega\neq 1\,.$$

Example 4. If g is an entire function with $g(0)=0$, then the function $f(z)=g\left((z-\omega)^{-1}\right)$ is analytic in $\overline{\mathbb{C}}\setminus\{\omega\}$, and we obtain

$$(6.6)\qquad \sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}=f(z)\qquad (\text{Ex})\ ,\qquad \text{for } z\neq\omega\,,$$

an expansion that is convergent for $|z|>|\omega|$. The coefficients a_n in this development are the same coefficients in the convergent Taylor expansion

$$(6.7)\qquad h(\xi)=g\left(\frac{\xi}{1-\xi\omega}\right)=\sum_{n=0}^{\infty}a_n\xi^{n+1}\,,\qquad |\xi|<|\omega|^{-1}\,,$$

so that $a_n=h^{(n+1)}(0)/(n+1)!$. Notice that the series $\sum_{n=1}^{\infty}a_n\xi^{n+1}$ is actually (Ex') summable to $h(\xi)$ for all $\xi\neq 1/\omega$.

The example considered in the proof of the Theorem 4.3, namely,

$$(6.8)\qquad f(z)=z^2e^{-1/z}-z^2+z-\frac{1}{2}\,,$$

$$(6.9)\qquad \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(n+3)!z^{n+1}}=f(z)\qquad (\text{Ex})\ ,\qquad z\neq 0\,,$$

$$(6.10)\qquad \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(n+3)!z^{n+1}}=-\frac{1}{2}\qquad (\text{Ex})\ ,\qquad z=0\,,$$

shows that when g is not a polynomial, it is possible for the series to be exterior Euler summable *everywhere.* Actually, a series that is exterior Euler summable everywhere can only arise in this way. In the case of (6.10) the series $\sum_{n=0}^{\infty}a_{n,\lambda}/(z+\lambda)^{n+1}$ converges, to $-1/2$, for $z=0$ whenever $\lambda>0$, because, in fact, the extension of $f(z)$ to the circle $|z+\lambda|\geq\lambda$ obtained by assigning the value $-1/2$ to $z=0$, is continuous in $\mathbb{C}\setminus D_\lambda$, while on the circle $|z+\lambda|=\lambda$ this extension, namely $f\left(-\lambda+\lambda e^{i\theta}\right)$, is a differentiable function of θ, even at $\theta=0$. Interestingly, if $\lambda<0$ then $f\left(-\lambda+\lambda e^{i\theta}\right)$ is also a differentiable function of θ, but the series $\sum_{n=0}^{\infty}a_{n,\lambda}/(z+\lambda)^{n+1}$ does not converge when $z=0$.

Example 5. Consider the series

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)\,z^{n+1}}=\ln\left(\frac{z}{z-1}\right)\,,\qquad |z|>1\,. \tag{6.11}$$

Here we obtain that $K_{\mathrm{cv}}=[0,1]$ and that the series is exterior Euler summable for any z that is not in the interval $[0,1]$. This, in turn, yields the formula

$$\sum_{n=1}^{\infty}\frac{\omega^{n}}{n}=\ln\left(\frac{1}{1-\omega}\right)\qquad (\mathrm{Ex'})\,,\qquad \text{for } \omega\notin[1,\infty)\,. \tag{6.12}$$

7. Convergence Acceleration

It is important to emphasize that the exterior Euler summability is not only a summability method, but actually it is a devise for the convergence acceleration of slowly convergent series, or even divergent series which are Cesàro or Abel summable.

Let us consider an Abel summable series, which we write in the form $\sum_{n=0}^{\infty}(-1)^{n+1}a_n$. Let S be the sum of the series,

$$\sum_{n=0}^{\infty}(-1)^{n+1}a_n=S\qquad (\mathrm{A})\,. \tag{7.1}$$

Our aim is to find a rapidly convergent representation of S.

The Abel summability imply that the function

$$f(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^{n+1}}\,, \tag{7.2}$$

is analytic in the region $\{z\in\mathbb{C}:|z|\geq 1\}$. Let $T=\mathcal{C}^{-1}\{f\}$, an analytic functional in the closed unit disc $\overline{\mathbb{D}}$. *Our key assumption* is that T admits an extension to $\mathfrak{O}'(K)$, where K is a compact convex subset of $\overline{\mathbb{D}}$ such that $-1\notin K$. If we take K minimal with this property, then $K=K_{\mathrm{cv}}$, f admits an analytic continuation f_{cv} to $\overline{\mathbb{C}}\setminus K_{\mathrm{cv}}$, a region that contains the point $z=-1$, and

$$S=f_{\mathrm{cv}}(-1)\,. \tag{7.3}$$

Observe that, in general, the series in (7.1) is not convergent, not even Cesàro summable. However, if $a_n=O\left(n^{\beta}\right)$ for some $\beta\geq 0$, then the series should be Cesàro summable of some order since in that case the Fourier series $\sum_{n=0}^{\infty}a_ne^{-in\theta}$ is a distribution in $\mathcal{D}'(\partial\mathbb{D})$ [2, 3, 6], and since f_{cv} is analytic at -1, this Fourier series represents a continuous function in a neighborhood of -1 in the circle, and thus the series is Cesàro summable in that neighborhood [5, 19].

Nevertheless, whether the series (7.1) is convergent or not, then our assumptions yield that

$$\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} = S \qquad (\mathrm{Ex}) , \quad \text{if } \ z = -1 , \tag{7.4}$$

so that there exist complex numbers λ such that

$$S = \sum_{n=0}^{\infty} \frac{a_{n,\lambda}}{(-1+\lambda)^{n+1}} , \tag{7.5}$$

where

$$a_{n,\lambda} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} a_j , \tag{7.6}$$

is a convergent series. Actually, if $-\lambda \in (0, \infty)$, the series (7.5) is *exponentially convergent.* If a power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, and $|z| < R$, we say that the series converges like $(|z| / R)^n$; if $R = \infty$ we say that the series converges like an entire function. For example, if $K_{\mathrm{cv}} = [0, 1]$, by taking $-\lambda = 1/2$ then the series converges like $(1/3)^n$, while if $-\lambda = 1$ then the convergence is like $(1/2)^n$. See [1] for an analysis of the best way to choose λ in order to minimize the error when using a partial sum, with a fixed number of terms, of the series (7.5); [1] also has several very interesting numerical evaluations of series.

Let us illustrate this procedure with the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$, which is Abel summable for *all* values of $s \in \mathbb{C}$; it converges when $\Re e \, s > 0$. We have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1 - 2^{s-1}\right) \zeta(s) \qquad (\mathrm{A}) , \tag{7.7}$$

where $\zeta(s)$ is the Riemann zeta function. The convergence acceleration of $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$ thus provides a procedure for the numerical evaluation of $\zeta(s)$ (the numerical evaluation when $s > 0$ is given in [1]).

In this case there exist distributions $T_s(x)$ for $s \in \mathbb{C}$, with Cauchy representations $f_s(z)$, such that

$$2\pi i f_s(-1) = \left\langle T_s(x) , \frac{1}{x+1} \right\rangle = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} . \tag{7.8}$$

As we shall see, $\operatorname{supp} T_s = [0,1]$ for $s \in \mathbb{C} \setminus \{0,1,2,\ldots\}$, while for $q = 0,\, -1,\, -2,\, \ldots$ we have $\operatorname{supp} T_{-q} = \{1\}$. In order to construct the distributions $T_s(x)$ we use the well known formula

$$\int_0^\infty t^{s-1} e^{-kt} \mathrm{d}t = \frac{\Gamma(s)}{k^s}\,, \tag{7.9}$$

and make a change of variables to obtain

$$\int_0^1 x^n \ln^{s-1}(1/x)\, \mathrm{d}x = \frac{\Gamma(s)}{(n+1)^s}\,. \tag{7.10}$$

The function $\widetilde{T}_s(x) = \chi_{(0,1)}(x) \ln^{s-1}(1/x)$ is locally integrable in $\mathbb{R} \setminus \{1\}$, and at $x = 1$ it behaves like $\widetilde{T}_s(x) \sim (1-x)^{s-1}$ as $x \to 1^-$. It follows that $\widetilde{T}_s(x)$ defines a distribution for $s \neq 0, -1, -2, \ldots$, analytic as a function of s, with simple poles at the negative integers. Therefore, if we define

$$T_s(x) = \frac{\widetilde{T}_s(x)}{\Gamma(s)}\,, \tag{7.11}$$

then the distribution $T_s(x)$ is an entire function of s, with moments

$$\mu_{s,n} = \langle T_s(x), x^n \rangle = \frac{1}{(n+1)^s}\,. \tag{7.12}$$

Our results give the exterior Euler expansions

$$\sum_{n=1}^\infty \frac{1}{n^s z^n} = 2\pi i f_s(z) \qquad (\mathrm{Ex})\,, \qquad \text{for } \ z \notin [0,1]\,, \tag{7.13}$$

if $s \neq 0,\, -1,\, -2,\, \ldots$, and

$$\sum_{n=1}^\infty \frac{n^q}{z^n} = 2\pi i f_{-q}(z) \qquad (\mathrm{Ex})\,, \qquad \text{for } \ z \neq 1\,, \tag{7.14}$$

if $q = 0,\, -1,\, -2,\, \ldots$. If we now take $z = -1$ and use the scheme given by (7.5) and (7.6) we obtain

$$\left(1 - 2^{1-s}\right) \zeta(s) = -\sum_{n=0}^\infty \sum_{j=0}^n \binom{n}{j} \frac{\lambda^{n-j}}{(-1+\lambda)^{n+1} (1+j)^s}\,, \tag{7.15}$$

(that reduces to formula (19) of [1] if we replace λ by $-\lambda$). If $\lambda = -1$ we obtain the formula [12, 18]

$$\left(1 - 2^{1-s}\right) \zeta(s) = \sum_{n=0}^\infty \frac{1}{2^{n+1}} \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{(1+j)^s}\,, \tag{7.16}$$

that converges like $(1/2)^n$ if $s \neq 0, -1, -2, \ldots$, and that reduces to a finite sum in case $s = 0, -1, -2, \ldots$. If $-\lambda = 1/2$, then the convergence in (7.15) is like $(1/3)^n$ for any $s \in \mathbb{C}$.

8. Mittag-Leffler Expansions

We shall now consider the exterior Euler sum representation of some Mittag-Leffler expansions.

Let us start with a series $\sum_{n=1}^{\infty} a_n$ that is Cesàro summable [9, 11]. Then the series of distributions

$$T(x) = \sum_{n=1}^{\infty} a_n \delta\left(x - \frac{1}{n}\right) , \tag{8.1}$$

is Cesàro summable in the space $\mathcal{D}'(\mathbb{R})$ (actually in the space $\mathcal{E}'(\mathbb{R})$). Indeed, if $\sum_{n=1}^{\infty} a_n$ is (C) summable, then so is $\sum_{n=1}^{\infty} a_n n^{-\beta}$ for any $\beta > 0$. Also, there exists $N \in \mathbb{N}$ such that $a_n = O\left(n^N\right)$ as $n \to \infty$. If $\phi \in \mathcal{E}(\mathbb{R})$ then we can write

$$\phi(x) = \sum_{j=0}^{N+1} \frac{\phi^{(j)}(0)\, x^j}{j!} + x^{N+2}\psi(x) , \tag{8.2}$$

for some function $\psi \in \mathcal{E}(\mathbb{R})$, and thus we obtain

$$\begin{aligned}
\langle T(x), \phi(x)\rangle &= \sum_{n=1}^{\infty} a_n \phi\left(\frac{1}{n}\right) \\
&= \sum_{n=1}^{\infty} a_n \left\{\phi(0) + \frac{\phi'(0)}{n} + \cdots \frac{\phi^{(N+1)}(0)}{(N+1)!\, n^{N+1}} + \frac{\psi\left(\frac{1}{n}\right)}{n^{N+2}}\right\} \\
&= \sum_{j=0}^{N+1} \sum_{n=1}^{\infty} \frac{a_n}{n^j} \frac{\phi^{(j)}(0)}{j!} + \sum_{n=1}^{\infty} \frac{a_n}{n^{N+2}} \psi\left(\frac{1}{n}\right) \qquad (\mathrm{C}) ,
\end{aligned}$$

as the sum of $N+1$ Cesàro summable series and a convergent one.

More generally, for a series of the type $\sum_{n=-\infty}^{\infty} a_n$ that is Cesàro summable in the *principal value* sense at infinity [9], namely, if the *symmetric* Cesàro limit $\sum_{n=-N}^{N} a_n$ exists as $N \to \infty$, then the series $\sum_{n=-\infty, n\neq 0}^{\infty} a_n \delta(x - 1/n)$ is likewise principal value Cesàro summable in the space $\mathcal{E}'(\mathbb{R})$.

Observe that the analytic representation of a distribution of the type

$$T_\alpha(x) = \text{p.v.} \sum_{n=-\infty, n\neq 0}^{\infty} a_n \delta\left(x - \frac{\alpha}{n}\right) \qquad (\mathrm{C}) , \tag{8.3}$$

for $\alpha \in \mathbb{R} \setminus \{0\}$ is given by

$$2\pi i f_{\alpha}(z) = \left\langle T_{\alpha}(x), \frac{1}{x-z} \right\rangle = \text{p.v.} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{n a_n}{\alpha - n z} \quad \text{(C)}\ , \tag{8.4}$$

for $z \in \mathbb{C} \setminus \operatorname{supp}(T_{\alpha})$. If we now put $b_n = n a_n$, and $\omega = \alpha / z$, we obtain the following result.

Lemma 8.1. *If the series* $\sum_{n=-\infty, n\neq 0}^{\infty} b_n / n$ *is principal value Cesàro summable, then the series*

$$G(\omega) = \text{p.v.} \sum_{n=-\infty}^{\infty} \frac{b_n}{\omega - n} \quad (C)\ , \tag{8.5}$$

is also principal value Cesàro summable for all $\omega \in \mathbb{C} \setminus \mathbb{Z}$, G *is analytic in this region and has single poles at the integers* $n \in \mathbb{N}$, *with residues* b_n.

Observe that the sum in (8.5) may have a term corresponding to $n = 0$. The results of Section 5 yield the following exterior Euler series representation of the Mittag-Leffler function G.

Theorem 8.2. *The analytic function* G *given by (8.5) can be written as the exterior Euler summable series*

$$G(\omega) = \frac{b_0}{\omega} - \sum_{k=0}^{\infty} \xi_k \omega^k \quad (Ex')\ , \tag{8.6}$$

for $\omega \neq 0$, $\omega \notin (-\infty, -1] \cup [1, \infty)$, *where the moments are given as*

$$\xi_k = \text{p.v.} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{b_n}{n^{k+1}} \quad (C)\ . \tag{8.7}$$

Actually one can give a related expansion which is valid in the region $\mathbb{C} \setminus ((-\infty, -N-1] \cup [N+1, \infty))$, $\omega \neq 0, \pm 1, \ldots, \pm N$, namely,

$$G(\omega) = \sum_{n=-N}^{N} \frac{b_n}{\omega - n} - \sum_{k=0}^{\infty} \xi_{k,N} \omega^k \quad \text{(Ex')}\ , \tag{8.8}$$

where

$$\xi_{k,N} = \text{p.v.} \sum_{n=-\infty, |n|>N}^{\infty} \frac{b_n}{n^{k+1}} \quad \text{(C)}\ . \tag{8.9}$$

An example is provided by $G(\omega) = \pi \cot \pi\omega$, that has the principal value convergent Mittag-Leffler expansion p.v. $\sum_{n=-\infty}^{\infty} 1/(\omega - n)$:

$$\pi \cot \pi\omega = \frac{1}{\omega} + 2\sum_{n=1}^{\infty} \zeta(2n)\,\omega^{2n-1} \qquad (\text{Ex'}) \ , \tag{8.10}$$

for any complex number $\omega \neq 0$ with $\omega \notin (-\infty, -1] \cup [1, \infty)$. Here ζ is the Riemann zeta function.

We can also consider Mittag-Leffler developments that are not principal value (C) summable. We shall illustrate this with an expansion for the digamma function $\psi(\omega) = \Gamma'(\omega)/\Gamma(\omega)$, whose Mittag-Leffler development is given by

$$\psi(\omega) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+\omega} \right) , \qquad \omega \neq 0, -1, -2, \ldots , \tag{8.11}$$

where γ is Euler's constant. In this case the series of analytic functionals $\sum_{n=1}^{\infty}(1/n)\delta(\omega - \alpha/n)$ is not (C) summable for any $\alpha \in \mathbb{C}$, but the analytic functional

$$T(\omega) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\delta\left(\omega - \frac{\alpha}{n}\right) - \delta\left(\omega - \frac{\beta}{n}\right) \right) , \tag{8.12}$$

is given by a convergent series for any $\alpha, \beta \in \mathbb{C}$. The moments are

$$\mu_k = \left\langle T(\omega), \omega^k \right\rangle = \left(\alpha^k - \beta^k\right) \zeta(k+1) \ , \quad k \in \mathbb{N} , \tag{8.13}$$

while its Cauchy representation $f = \mathcal{C}\{T\}$ is given by

$$\begin{aligned} 2\pi i f(z) &= \left\langle T(z), \frac{1}{\omega - z} \right\rangle \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{\alpha - nz} - \frac{1}{\beta - nz} \right) \\ &= \frac{1}{z} \left(\psi(1 - \alpha/z) - \psi(1 - \beta/z) \right) . \end{aligned} \tag{8.14}$$

If we now employ the Theorem 5.1, taking into account that $\mu_0 = 0$, we obtain

$$\sum_{k=1}^{\infty} \frac{\left(\alpha^k - \beta^k\right) \zeta(k+1)}{z^k} = \psi\left(1 - \frac{\beta}{z}\right) - \psi\left(1 - \frac{\alpha}{z}\right) \qquad (\text{Ex}) \ , \tag{8.15}$$

as long as $z \notin K(\alpha, \beta)$, where the triangular set $K(\alpha, \beta)$ is the smallest convex set that contains α, β, and 0, that is, the convex support of the analytic functional T.

When $1 \notin K(\alpha, \beta)$ then (8.15) yields

$$\sum_{k=1}^{\infty} \left(\alpha^k - \beta^k\right) \zeta(k+1) = \psi(1-\beta) - \psi(1-\alpha) \qquad (\text{Ex'}) \, . \tag{8.16}$$

If we take $\alpha = 0$ and use the fact that $\psi(1) = -\gamma$ we obtain

$$\sum_{k=1}^{\infty} \beta^k \zeta(k+1) = -\psi(1-\beta) - \gamma \qquad (\text{Ex'}) \, , \qquad \beta \notin [1, \infty) \, . \tag{8.17}$$

In particular, if $\beta = -1$ we find the sum of the not only exterior Euler summable but actually Cesàro summable series,

$$\sum_{k=1}^{\infty} (-1)^{k+1} \zeta(k+1) = 1 \qquad (\text{C}) \, . \tag{8.18}$$

When $\beta = -N$, $N = 2, 3, 4, \ldots$, we obtain exterior Euler summable series that are not Abel summable,

$$\sum_{k=1}^{\infty} (-1)^{k+1} N^k \zeta(k+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{N} \qquad (\text{Ex'}) \, . \tag{8.19}$$

If we now take $\alpha = 1 - \omega$, $\beta = \omega$, and use the identity $\psi(1-\omega) - \psi(\omega) = \pi \cot \pi\omega$, then (8.16) yields that for $\omega \neq 0$, $\omega \notin (-\infty, -1] \cup [1, \infty)$,

$$\sum_{k=1}^{\infty} \left((1-\omega)^k - \omega^k\right) \zeta(k+1) = \pi \cot \pi\omega \qquad (\text{Ex'}) \, . \tag{8.20}$$

If $\omega = 1/4$ the series becomes convergent, and we recover the Flajolet-Vardi formula [10]

$$\sum_{k=1}^{\infty} \left(\left(\frac{3}{4}\right)^k - \left(\frac{1}{4}\right)^k\right) \zeta(k+1) = \pi \, , \tag{8.21}$$

considered also by Amore [1, Eqn. 4].

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