Functional Determinants on Riemann Caps

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Background on Geometry

The Riemann Cap (or Spherical Suspension) is defined as the $D = d + 1$ dimensional compact manifold $\Omega = \mathcal{I} \times \mathscr{N}$, with $\mathcal{I} \subseteq [0, \theta_0]$, and where $\mathscr{N}$ represents a smooth, compact Riemannian d-dimensional base manifold. Ω is locally described by the line element

$$ds^2 = d\theta^2 + \sin^2\theta d\Sigma^2_{\mathscr{N}} \ .$$

Remarks:

- Generalization of the spherical caps $ds^2_D = d\theta^2 + \sin^2\theta ds^2_{D-1}$.
- Presence of a conical singularity for θ "small".

We consider the Laplace operator Δ_Ω acting on scalar functions $\varphi \in \mathscr{L}^2(\Omega)$. In hypershperical coordinates

$$\Delta_\Omega = \frac{\partial^2}{\partial\theta^2} + d\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{\mathscr{N}} \ ,$$

with $\Delta_{\mathscr{N}}$ being the Laplacian on $\mathscr{N}$.

Eigenvalues and Eigenfunctions

The eigenvalue equation

$$\left(-\Delta_\Omega + m^2\right)\varphi = \alpha^2\varphi \ ,$$

is separable and has a solution of the form

$$\varphi(\theta, X_j) = (\sin\theta)^{\frac{(1-d)}{2}}\psi(\theta)\mathcal{H}(X_j) \ , \qquad \Delta_{\mathscr{N}}\mathcal{H}(X_j) = -\lambda^2\mathcal{H}(X_j) \ ,$$

with $\mathcal{H}(X_j)$ being the hypershperical harmonics on $\mathscr{N}$.
The specific solution of the eigenvalue equation which is regular for $\theta \to 0$ is

$$\varphi = (\sin\theta)^{\frac{(1-d)}{2}} P^{-\mu}_{-\frac{1}{2}+i\omega}(\cos\theta)\,\mathcal{H} \ ,$$

$$\mu = \sqrt{\frac{(1-d)^2}{4} + \lambda^2} \ , \quad -\frac{1}{2} + i\omega \equiv -\frac{1}{2} + i\sqrt{\alpha^2 - \sigma^2} \ ,$$

where $\sigma^2 = m^2 + d^2/4$.

Spectral Zeta Function

The *spectral zeta function* of the problem is defined as

$$\zeta(s) = \sum_{\alpha} \alpha^{-2s} = \sum \left(\omega^2 + \sigma^2\right)^{-s} \ .$$

Since the base manifold is unspecified, we will express $\zeta(s)$ in terms of the spectral zeta function on the base $\zeta_{\mathcal{N}}(s)$ defined as

$$\zeta_{\mathcal{N}}(s) = \sum_{\mu} d(\mu)\mu^{-2s} \ .$$

Here, we are interested in imposing Dirichlet boundary conditions at $\theta = \theta_0$. This leads to

$$P^{-\mu}_{-\frac{1}{2}+i\omega}(\cos\theta_0) = 0 \ .$$

The above equation implicitly determines the eigenvalues ω (and therefore α).

Zeta Function and Functional Determinants

Why do we care about spectral zeta functions?

- The spectral zeta function is a powerful tool for the evaluation of the *regularized* functional determinant of an elliptic operator $\mathcal{L}$,

$$\mathrm{Det}\mathcal{L} \equiv e^{-\zeta'(0)} .$$

- The spectral zeta function is utilized in order to compute the One-Loop effective action Γ in quantum field theory,

$$\Gamma = -\frac{1}{2}\zeta'(0) - \frac{1}{2}\zeta(0)\ln\mu^2 ,$$

  where μ is a parameter with the dimensions of mass.

There are many other applications of the spectral zeta function in Mathematical Physics, Spectral Geometry and Casimir Effect.

Integral Representation of $\zeta(s)$

The starting point of our analysis is the following representation

$$\zeta(s) = \sum_{\mu} d(\mu) \frac{1}{2\pi i} \int_{\gamma} dz \left(z^2 + \sigma^2\right)^{-s} \frac{\partial}{\partial z} \ln \mathrm{P}^{-\mu}_{-1/2+iz}(\cos\theta_0) \ .$$

By deforming the contour of integration to the imaginary axis and by setting $\mu^2 u = z^2 - \sigma^2$ we have

$$\zeta(s) = \frac{\sin \pi s}{\pi} \int_0^{\infty} \frac{du}{u^s} \frac{\partial \mathcal{G}}{\partial u}(u, s) \ ,$$

where

$$\mathcal{G}(u, s) = \sum_{\mu} d(\mu) \mu^{-2s} \ln \mathrm{P}^{-\mu}_{-1/2+\sqrt{u\mu^2+\sigma^2}}(\cos\theta_0) \ .$$

Notice that the above representations are valid for $\Re[s] > D/2$ *!*

Analytic Continuation

The next step is to analytically continue $\zeta(s)$ in the neighborhood of $s = 0$. To this end, an important result is available

Lemma

Let $f(x)$ be a function defined for $x \geq \epsilon$ with $\epsilon > 0$ and analytic at $x = \epsilon$. Assume that $f(x)$ has the following general asymptotic behavior for $x \to \infty$:

$$f(x) = \sum_{k=1}^{\rho_k < N} \left(f_k + \bar{f}_k \ln x \right) x^{\rho_k} + [f]_{\mathrm{log}} \ln x + [f]_{\mathrm{reg}} + O(x^{-1}) \,, \quad \rho_k > 0$$

where the subscripts log and reg refer to the solely logarithmic and regular (non-singular) parts of $f(x)$ in the large x limit. Then, there exists the analytic continuation of the integral

$$\int_{\epsilon}^{\infty} \frac{dx}{x^s} \frac{d}{dx} f(x) = \frac{[f]_{\mathrm{log}}}{s} + [f]_{\mathrm{reg}} - f(\epsilon) + O(s) \,.$$

Analytic Continuation

Why is this Lemma useful?

In general the function $\mathcal{G}(u,s)$ is well defined for $\Re[s] > s_0$ and its analytic continuation to $s = 0$ will have the form

$$\mathcal{G}(u,s) = \frac{1}{s}\mathcal{G}_P(u) + \mathcal{G}_R(u) + O(s) \ .$$

By assuming that $\mathcal{G}(u,s)$ satisfies the hypothesis of the Lemma, we obtain

$$\begin{aligned}\zeta(s) \quad &\sim \quad \frac{[\mathcal{G}_P]_{log}}{s} + [\mathcal{G}_R]_{log} + [\mathcal{G}_P]_{reg} - \mathcal{G}_P(0) \\ &+ \quad s\left([\mathcal{G}_R]_{reg} - \mathcal{G}_R(0) - \int_0^\infty du \ln u \frac{d}{du}\mathcal{G}_P(u)\right) \ .\end{aligned}$$

$\zeta(0)$ and $\zeta'(0)$

The previous relation gives the following results

$$\boxed{\begin{aligned} \zeta(0) &= [\mathcal{G}_R]_{log} + \Big([\mathcal{G}_P]_{reg} - \mathcal{G}_P(0)\Big) \ , \\ \zeta'(0) &= \Big([\mathcal{G}_R]_{reg} - \mathcal{G}_R(0)\Big) - \int_0^\infty du \ln u \tfrac{d}{du}\mathcal{G}_P(u) \ . \end{aligned}}$$

Where we have assumed that $[\mathcal{G}_P]_{log} = 0$ in order to have a *well defined* functional determinant.

Remark:

One can prove that

$$[\mathcal{G}_P]_{log} = -\frac{1}{2}\mathrm{Res}\,\zeta_{\mathcal{N}}\left(-\frac{1}{2}\right) = \mathrm{Res}\,\zeta(0) \ .$$

So if $\mathcal{N}$ is closed $[\mathcal{G}_P]_{log} = 0$ in even dimensions.

Riemann Caps Case

Can we apply the Lemma to our case?

The logarithm of the Legendre functions has the following asymptotic expansion for $\mu \to \infty$

$$\ln P^{-\mu}_{-1/2+\sqrt{u\mu^2+\sigma^2}}(\cos\theta_0) \sim \frac{1}{2}\ln t(u) - \frac{1}{2}\ln 2\pi\mu + \mu\tau(u) - \mu\ln\mu u + \sum_{n=0}^{\infty}\mu^{-n} a_n(t(u)\cos\theta_0) \,.$$

where the functions a_n are given by a recursion relation.

From here one can see that the asymptotic behavior required by the lemma is exactly reproduced.

Therefore, we can apply the Lemma to the function $\mathcal{G}(u,s)$ of our integral representation of $\zeta(s)$.

Computation of the Needed Terms

The only thing left to do is to evaluate all the terms needed in the expression for $\zeta(0)$ and $\zeta'(0)$.

Essential steps:

1. Identify the *logarithmic* and *regular* parts of the asymptotic expansion of $\ln P^{-\mu}_{-1/2+\sqrt{u\mu^2+\sigma^2}}$ for $\mu \to \infty$. These terms will have the general form

$$\begin{aligned} [\ln P]_{log} &\sim \mu^{\alpha} C_1(\sigma, \theta_0) + \ln \mu \, C_2(\sigma, \theta_0) \, , \\ [\ln P]_{reg} &\sim \mu^{\beta} C_3(\sigma, \theta_0) + \ln \mu \, C_4(\sigma, \theta_0) \, . \end{aligned}$$

2. Perform the sum over the eigenvalues μ. The result will be expressed in terms of $\zeta_{\mathcal{N}}(s)$ and $\zeta'_{\mathcal{N}}(s)$.
3. Utilize the analytic structure of $\zeta_{\mathcal{N}}(s)$ near $s = 0$ in order to find $[\mathcal{G}_P]_{log}$, $[\mathcal{G}_P]_{reg}$, $[\mathcal{G}_R]_{log}$ and $[\mathcal{G}_R]_{reg}$.

Evaluation of $\mathcal{G}_P(u)$, $\mathcal{G}_P(0)$ and $\mathcal{G}_R(0)$

The terms that are left to evaluate can be obtained by using the following procedure:

- $\mathcal{G}_P(u)$ is obtained by expanding in inverse powers of μ the asymptotic expansion of $\ln P$ and considering *only* the polar part in s.

$$\mathcal{G}_P(u) = \sum_{n=1}^{d} a_n(t(u)\cos\theta_0)\mathrm{Res}\,\zeta_{\mathcal{N}}\left(\frac{n}{2}\right)\;, \quad \mathcal{G}_P(0) = \lim_{u\to 0}\mathcal{G}_P(u)$$

- $\mathcal{G}_R(0)$ is obtained from the expression

$$\mathcal{G}_R(0) = \mathrm{PF}\left\{\lim_{s\to 0}\sum_{\mu} d(\mu)\mu^{-2s}\ln P^{-\mu}_{\sigma-1/2}(\cos\theta_0)\right\}\;,$$

which can be dealt with by first writing P in terms of a hypergeometric and then using the Abel-Plana summation formula.

Results

For the Riemann Cap we have obtained the following results:

- We have found a general formula for $\zeta(0)$ and $\zeta'(0)$ in arbitrary dimensions for an arbitrary smooth and compact base manifold $\mathscr{N}$ in terms of $\zeta_{\mathscr{N}}(s)$ and $\zeta'_{\mathscr{N}}(s)$.
- As a particular case we have assumed $\mathscr{N}$ to be a d-dimensional sphere. In this case $\zeta_{\mathscr{N}}(s)$ becomes a Barnes zeta function and more explicit results for $\zeta(0)$ and $\zeta'(0)$ in arbitrary dimensions have been given.
- In the case of a d-dimensional sphere as base manifold we have obtained specific expressions for $\zeta(0)$ and $\zeta'(0)$ for $D = 3, 4, 5$.

Final Remarks

The method used has its limitations

- *Pros.* This formulation gives the value of $\zeta(0)$ and $\zeta'(0)$ in a very direct way in terms of specific parts of the asymptotic expansion of the eigenfunctions.
- *Cons.* This method *only* provides the analytic continuation of $\zeta(s)$ at $s = 0$. Therefore it is not suitable for the evaluation, for instance, of the Heat Kernel coefficients and the Casimir Energy.

In a joint work with K. Kirsten we were able, by using a different method, to evaluate the Heat Kernel coefficients for Laplace operators on the Riemann Cap.

Open Problems

The computation of the Casimir Energy on the Riemann Cap would be a subject of interest. One could proceed in two ways:

- Generalize the method described here in order to find the analytic continuation at $s = -1/2$. The hope is to find an expression for $\zeta(-1/2)$ similar to the ones obtained for $\zeta(0)$ and $\zeta'(0)$.
- Analytically continue $\zeta(s)$ at $s = -1/2$ by adding and subtracting N leading terms of the asymptotic expansion of the eigenfunctions (often used in recent literature). However, in order to make the resulting integrals manageable, one would be forced to deal with the following spectral function

$$\zeta_{d^2/4}(s) = \sum_{\alpha} \left(\alpha^2 + \frac{d^2}{4} \right)^{-s} .$$

References

- (with A. Flachi)
  Zeta Determinant for Laplace Operators on Riemann Caps,
  arXiv:math-ph 1004.0063 (Under Review)
- (with K. Kirsten)
  Heat Kernel Coefficients for Laplace Operators on the Spherical Suspension,
  (coming soon)