The Power Wall

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Motivation

- Physical understanding requires $\langle T^{\mu\nu}(\mathbf{r})\rangle$.
- Perfectly reflecting wall $\Rightarrow$ divergence.
- Ultraviolet cutoff $\Rightarrow$ pressure paradox:

$$\frac{\partial E}{\partial h} \neq -\int_{S_h} p.$$

$$\frac{\partial E}{\partial x} = +\int_{S_x} p \quad \text{in dimension 2, intersecting planes.}$$

$$\frac{\partial E}{\partial h} = +2 \int_{S_h} p \quad \text{in dimension 3}$$

(for both intersecting planes and sphere).

- This shouldn't happen in an internally consistent dynamical theory.
- A steeply rising smooth potential
  - ∗ mocks up a reflecting wall.
  - ∗ should define a nonsingular, internally consistent theory.
- In principle this can be done for a spherical boundary, etc., but for now we do only a plane one.

Renormalization

Cylinder kernel divergences $\iff$ heat kernel.

In dimension 3, $\langle T^{00} \rangle$ should contain

- universal t^{-4} divergence
- t^{-2} term $\propto V(\mathbf{r})$
- $\ln t$ terms $\propto V^2$ and $\nabla^2 V$.

After removing these we should be able to set $t = 0$.
Only the first occurs where $V = 0$.

Elsewhere $\langle T^{\mu\nu} \rangle$ should include $V\varphi^2$ interaction term.

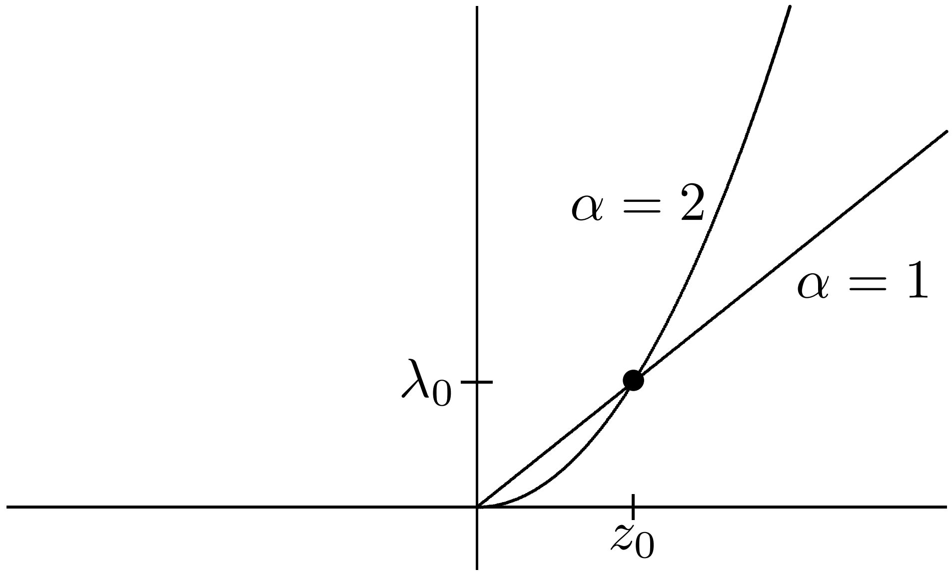
The model

$$v(x,y,z) = \begin{cases} 0, & z < 0 \\ \lambda z^\alpha, & z > 0. \quad (1 \le \alpha \in \mathbf{R}) \end{cases}$$

Get dimensions right: $v = \lambda_0 \left(\dfrac{z}{z_0}\right)^\alpha.$

Only one length scale: $\hat{z} = \lambda^{\frac{-1}{\alpha+2}} = \left(\dfrac{{z_0}^\alpha}{\lambda_0}\right)^{\frac{1}{\alpha+2}}.$

For any α, $v(z_0) = \lambda_0$; increasingly steep as $\alpha \to \infty$.



$\alpha = 2$
$\alpha = 1$
λ_0
z_0

$$\varphi(\mathbf{r}, x^0) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[a_n \phi_n(\mathbf{r}) e^{-i\omega_n x^0} + a_n^\dagger \phi_n(\mathbf{r}) e^{i\omega_n x^0} \right].$$

$$T^{00} = \lim_{t \to 0} -\frac{1}{2} \frac{\partial^2 \overline{T}}{\partial t^2}, \quad \text{etc.}$$

$$\overline{T}(t, \mathbf{r}, \mathbf{r}') = \sum_n \frac{\phi_n(\mathbf{r}) \phi_n(\mathbf{r}')}{-\omega_n} e^{-\omega_n t}.$$

$$\left(\frac{\partial^2}{\partial t^2} + \nabla^2 - v(z) \right) \overline{T}(t, x, y, z, z') = 2\delta(t)\delta(x)\delta(y)\delta(z - z').$$

Eigenfunctions

$$\phi_n(\mathbf{r}) = e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \phi_p(z)$$

$$(\mathbf{r}_\perp,\ \mathbf{k}_\perp \in \mathbf{R}^2, \quad z \in \mathbf{R},\ p \in (0,\infty).)$$

$$\left(-\frac{\partial^2}{\partial z^2} + v(z) - p^2 \right) \phi_p(z) = 0.$$

$$\phi_p(z) = \sqrt{\frac{2}{\pi}} \sin[pz - \delta(p)] \quad \text{when } z < 0.$$

When $z > 0$, $\qquad \phi_p(z) = C(p) P_\alpha \left(\frac{z}{\hat{z}}, (\hat{z}p)^2 \right)$,

$$\left(-\frac{d^2}{dz^2} + z^\alpha - E \right) P_\alpha(z, E) = 0, \qquad P_\alpha(+\infty, E) = 0.$$

$$P_1(z, E) \propto \mathrm{Ai}(z - E), \qquad P_2(z, E) \propto D_{\frac{1}{2}(E-1)}(\sqrt{2}\, z).$$

For hard wall at z_0, $P_\infty(z, E) \propto \sin[\sqrt{E}(z - z_0)]$.
(Henceforth usually $\hat{z} = 1$, $\sqrt{E} = p$ $(z_0 = 1 = \lambda_0)$.)

The solutions must match at $z = 0$:

$$\tan\big(\delta(p)\big) = -p \, \frac{P_\alpha(0, p^2)}{P'_\alpha(0, p^2)} \, .$$

$$C(p)^2 = \frac{2}{\pi} \, \frac{1}{P_\alpha(0, p^2)^2 + p^{-2} P'_\alpha(0, p^2)^2} \, .$$

Even for $P = \mathrm{Ai}$, these formulas are unpleasant.

Small p

When $p = 0$ the solution is known:

$$P_\alpha(z,0) = z^{1/2} K_{\frac{1}{\alpha+2}}\left(\frac{2}{\alpha+2}\, z^{\frac{\alpha+2}{2}}\right).$$

Perturbation expansion:

$$P_\alpha(z,E) = P_\alpha(z,0) + E P_\alpha^{(1)}(z) + \cdots .$$

$$\delta(p) = p(\alpha+2)^{\frac{2}{\alpha+2}}\, \Gamma\left(\frac{\alpha+3}{\alpha+2}\right) \Gamma\left(\frac{\alpha+1}{\alpha+2}\right)^{-1} + O(p^3).$$

Large p (WKB)

$$\phi_p(z) \sim [p^2 - v(z)]^{-\frac{1}{4}} \cos\left[\int_z^a \sqrt{p^2 - v(\tilde{z})}\, d\tilde{z} - \frac{\pi}{4}\right],$$

turning point $a = p^{2/\alpha}$.

$$\delta(p) = \int_0^a \sqrt{p^2 - v(z)}\, dz + \frac{\pi}{4} \mod \pi$$

$$= \frac{1}{\alpha}\, p^{1+2/\alpha} \mathrm{B}\left(\frac{3}{2}, \frac{1}{\alpha}\right) + \frac{\pi}{4}\,.$$

$\alpha = 1$ (Airy function):

$$\delta(p) \sim \begin{cases} p\,3^{2/3}\Gamma(\tfrac{4}{3})/\Gamma(\tfrac{2}{3}), & p \to 0, \\ \dfrac{2p^3}{3} + \dfrac{\pi}{4}\,, & p \to \infty. \end{cases}$$

$\alpha = 2$ (parabolic cylinder function):

$$\delta(p) \sim \begin{cases} 2p\,\Gamma(\tfrac{5}{4})/\Gamma(\tfrac{3}{4}), & p \to 0, \\ \dfrac{\pi p^2}{4} + \dfrac{\pi}{4}\,, & p \to \infty. \end{cases}$$

Cylinder kernel calculations

Recall

$$\overline{T}(t,\mathbf{r},\mathbf{r}') = \sum_n \frac{\phi_n(\mathbf{r})\phi_n(\mathbf{r}')}{-\omega_n}\, e^{-\omega_n t}.$$

$$\left(\frac{\partial^2}{\partial t^2} + \nabla^2 - v(z)\right)\overline{T}(t,\mathbf{r}_\perp,z,z') = 2\delta(t)\delta^{(2)}(\mathbf{r}_\perp)\delta(z-z').$$

$$\hat{T}(\omega,\mathbf{k}_\perp,p) = \frac{-2}{(2\pi)^{3/2}}\,\frac{\phi_p(z')}{\omega^2 + {k_\perp}^2 + p^2}\,.$$

Cartesian calculations

$$\overline{T}(t,\mathbf{r}_\perp,z,z') = -\frac{1}{2\pi}\int_0^\infty dp\,Y(s,p)\phi_p(z)\phi_p(z'),$$

$$Y(s,p) \equiv \frac{e^{-sp}}{s}\,, \qquad s \equiv \sqrt{t^2+|\mathbf{r}_\perp|^2}\,.$$

In potential-free region, $z<0$,

$$\overline{T} = -\frac{1}{\pi^2}\int_0^\infty dp\,Y(s,p)\sin\bigl(pz-\delta(p)\bigr)\sin\bigl(pz'-\delta(p)\bigr).$$

$$
\begin{aligned}
\overline{T} = & - \frac{1}{2\pi^2} \frac{1}{t^2 + {r_\perp}^2 + (z - z')^2} \\
& + \frac{1}{2\pi^2} \int_0^\infty dp\, Y(s,p) \cos\bigl(p(z+z') - 2\delta(p)\bigr) \\
& \equiv \overline{T}_{\text{free}} + \overline{T}_{\text{ren}} \,.
\end{aligned}
$$

Hard wall: $\delta(p) = z_0 p \;\Rightarrow\;$ (correctly)

$$
\overline{T}_{\text{ren}} = \frac{1}{2\pi^2} \frac{1}{t^2 + {r_\perp}^2 + (z + z' - 2z_0)^2} \,.
$$

The bad news:

$$\overline{T}_{\rm ren} = \frac{1}{2\pi^2} \int_0^\infty dp\, \frac{e^{-sp}}{s}\, \cos\bigl(p(z+z') - 2\delta(p)\bigr)$$

is poorly convergent when $s \equiv \sqrt{t^2 + {r_\perp}^2}$ is small, which is precisely where we want it. In fact, we should be able to take $s = 0$ and get a finite answer when $z + z' > 0$, but the integrand is pointwise infinite there!

There is a genuine divergence for $\delta(p) = Ap + B$ unless $B = 0$. Asymptotics for *small* p as well as large is critical. Presumably large $\mathbf{k}_\perp$ is at fault, so ...

Polar calculations

Do the integral in polar coordinates in the Fourier space: $\quad (Z \equiv z + z',\ \mathbf{s} = (t, \mathbf{r}_\perp),\ \mathbf{v} = (\omega, \mathbf{k}_\perp))$

$$\overline{T}_{\text{ren}} = \frac{1}{4\pi^4} \int_0^\infty dp \int_{\mathbf{R}^3} d\mathbf{v} \, \frac{e^{i\mathbf{v}\cdot\mathbf{s}}}{v^2 + p^2} \, \cos\bigl(pZ - 2\delta(p)\bigr)$$

$$= \frac{1}{\pi^3} \int_0^\infty d\rho \int_0^1 du \, s^{-1} \sin(s\rho\sqrt{1-u^2})$$

$$\times \cos\bigl(Z\rho u - 2\delta(\rho u)\bigr).$$

With $s = 0$ and $z = z'$,

$$\overline{T}_{\mathrm{ren}}(0,0,z,z) = \frac{1}{\pi^3}\int_0^\infty d\rho \int_0^1 du\, \rho\sqrt{1-u^2} \times \cos\bigl(2z\rho u - 2\delta(\rho u)\bigr).$$

Integration by parts (appealing to $\delta(0) = 0\,!$) improves the convergence but worsens the algebra:

$$\overline{T}_{\mathrm{ren}}(0,0,z,z) = -\frac{1}{2\pi^3}\int_0^\infty d\rho \int_0^1 du\, \frac{d}{du}\left[\frac{\sqrt{1-u^2}}{z-\delta'(\rho u)}\right] \times \sin\bigl(2z\rho u - 2\delta(\rho u)\bigr).$$

Numerics (BARELY STARTED)

- Padé interpolation between small and large p asymptotics works fairly well.
- The oscillatory integrals appear intractable so far, even with Riesz–Cesàro averaging.