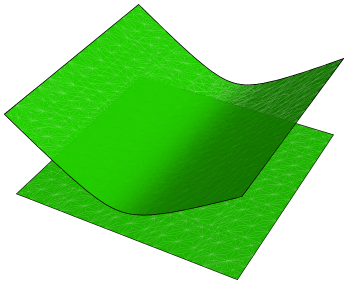
Casimir Self-Energies Revisited—Old and New Results

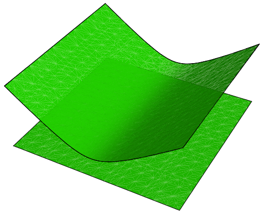
K. A. Milton

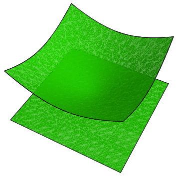
I. Cavero-Peláez (Zaragoza), Elom Abalo and J. Wagner (OU),

Klaus Kirsten (BU), Iver Brevik and Simen Ellingsen (NTNU)

supported by NSF and DOE







I. Quantum Vacuum Energy

- Important at all energy scales, subnuclear to cosmological
- Applications coming in nanotechnology
- Most likely the source of Dark Energy
  - 7-year WMAP data completely consistent with cosmological constant:

$$w = -1.10 \pm 0.14 (68\% \text{ CL})$$

- The most fundamental aspect of quantum field theory—Zero-point fluctuations

II. Self-energies: spheres & cylinder

Type	$E_{\text{Sphere}}a$	$\mathcal{E}_{\text{Cylinder}}a^2$	References
EM	0.04618	-0.01356	Boyer, DeRaad
D	0.002817	0.0006148	Bender,Gosdzinsl
$(\varepsilon-1)^2$	$\frac{23}{1536\pi}$	0	Brevik, Cavero
ξ^2	$\frac{5}{32\pi}$	0	Klich, Milton
δe^2	± 0.0009	0	Kitson, Kitson
λ^2/a^2	$\frac{1}{32\pi}$	0	Milton, Cavero

III. Self-energy for a ST sphere

We use the multiple scattering formalism to confirm the result for the self-stress on a single sphere of radius a using this formalism. We start from the general formula

$$E = \frac{i}{2\tau} \text{Tr} \ln \frac{1}{1 + G_0 V},$$

where

$$V(\mathbf{r}, \mathbf{r}') = \lambda \delta(r - a) \delta(\mathbf{r} - \mathbf{r}').$$

In strong coupling ($\lambda \to \infty$) this becomes a Dirichlet sphere.

Partial wave expansion

The free propagator in Euclidean space is:

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2+\zeta^2},$$

and the partial wave expansion of the plane wave is:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{lm} 4\pi i^l j_l(kr) Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{k}}).$$

G fn in spherical basis

Then, from the orthonormality of the spherical harmonics,

$$\int d\hat{\mathbf{k}} Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}) = \delta_{ll'}\delta_{mm'},$$

we obtain the representation

$$G_0(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\sum_{lm}\int_0^\infty \frac{dk\,k^2}{k^2+\zeta^2} j_l(kr) j_l(kr') Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}').$$

G_0V

Now we combine the representation for the free Green's function with the spherical potential to obtain

$$(G_0V)(\mathbf{r},\mathbf{r}') = \frac{2\lambda}{\pi}\delta(r'-a)\sum_{lm}\int_0^\infty \frac{dk\,k^2}{k^2+\zeta^2} j_l(ka)j_l(kr)$$
$$\times Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}').$$

When this, or powers of this, is traced (that is, $\mathbf{r}$ and $\mathbf{r}'$ are set equal, and integrated over), we obtain a poorly defined expression.

Point-split regulation

To regulate this, we assume $r \neq a$, for example, $r < a$. (This is a type of point-split regulation.) Then, because

$$j_l(ka) = \frac{1}{2}\left(h_l^{(1)}(ka) + (-1)^l h_l^{(1)}(-ka)\right),$$

while $j_l(kr) = (-1)^l j_l(-kr)$, we see that the k integration can be evaluated as ($r < a$)

$$\int_0^\infty \frac{dk\,k^2}{k^2+\zeta^2} j_l(ka) j_l(kr) = \frac{\pi}{a} K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|r).$$

Expression for Casimir energy

Thus, it is easily seen that an arbitrary power of G_0V has trace

$$\mathrm{Tr}\,(G_0V)^n = (\lambda a)^n \sum_{lm} \left(K_{l+1/2}(|\zeta|a)I_{l+1/2}(|\zeta|a)\right)^n,$$

and that therefore the total self-energy of the semitransparent sphere is given by the well-known expression ($x = |\zeta|a$)

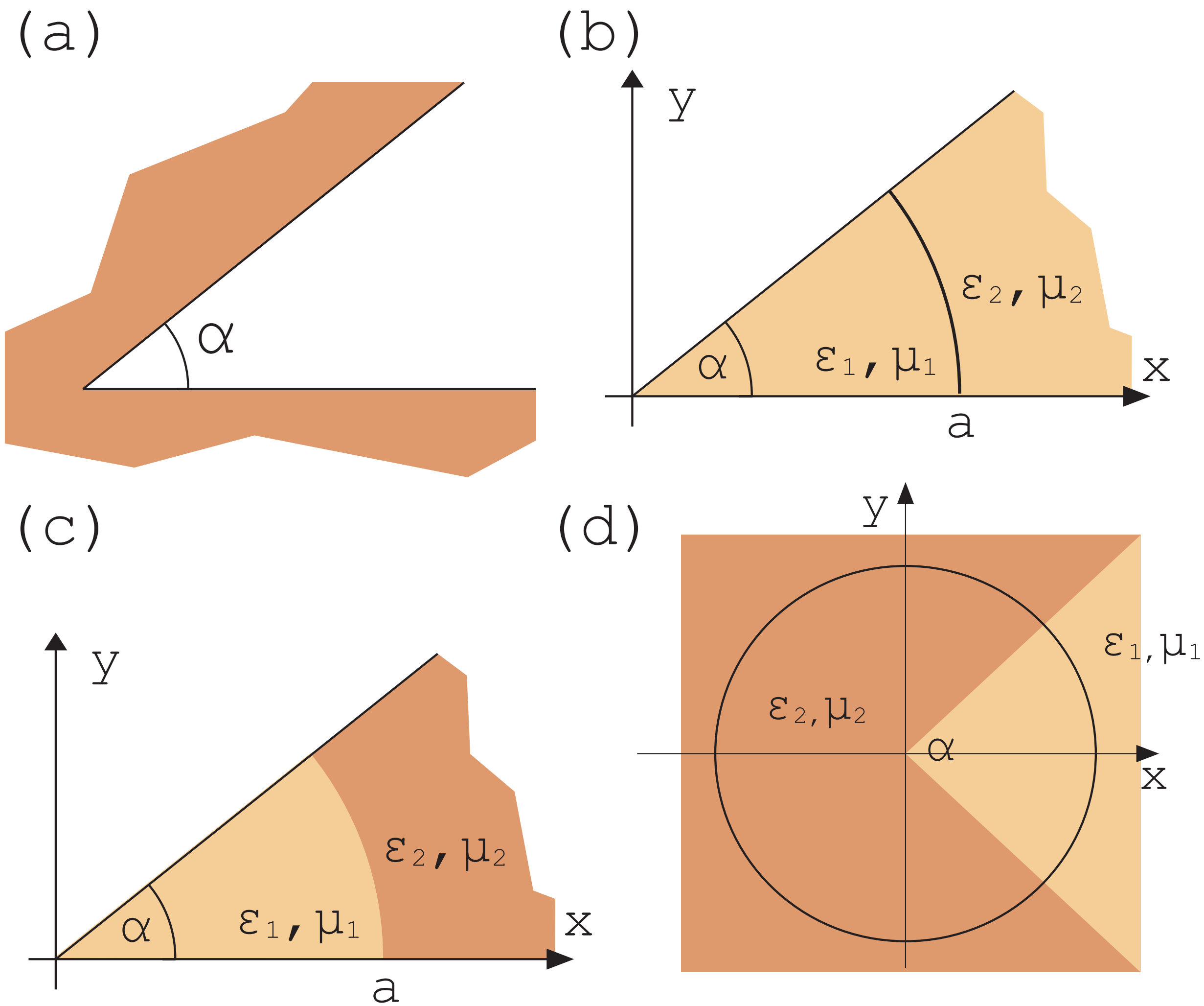
$$E = \frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dx\,\ln\left(1+\lambda a I_{l+1/2}(x)K_{l+1/2}(x)\right)$$

Finite results

Actually, a slightly different form involving integration by parts was given in our papers which results in the energy being finite though order λ^2 (result as shown in Table). In order λ^3 there is a divergence which is associated with surface energy.

The $\lambda \to \infty$ limit corresponds to a sphere on which Dirichlet boundary conditions are satisfied, and is evaluated to give the value stated above in the table.

IV. Wedge generalizes cylinder



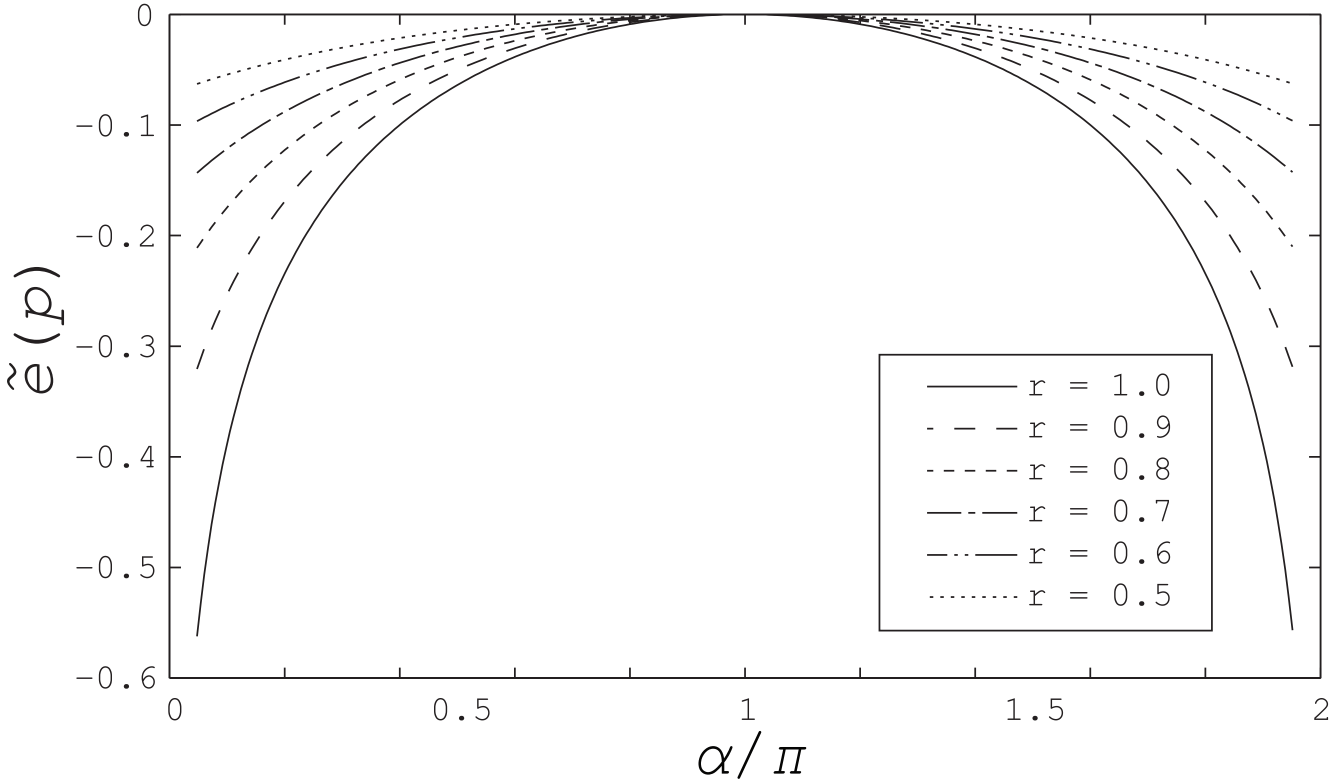
Case (d) dispersion relation

Using multiple scattering, or the Kontorovich-Lebedev transformation, we obtain the following implicit formula for the angular eigenvalues, (r=reflection coefficient on wedge)

$$\begin{aligned} D(\nu,\omega) &= (1 - e^{2\pi i\nu})^2 - r^2(e^{i\nu(2\pi-\alpha)} - e^{i\nu\alpha})^2 \\ &= -4e^{2\pi i\nu}[\sin^2(\nu\pi) - r^2\sin^2(\nu(\pi-\alpha))], \end{aligned}$$

which are selected by the "argument principle."

$\tilde{\mathcal{E}} = \frac{1}{8\pi n a^2} e(p)$

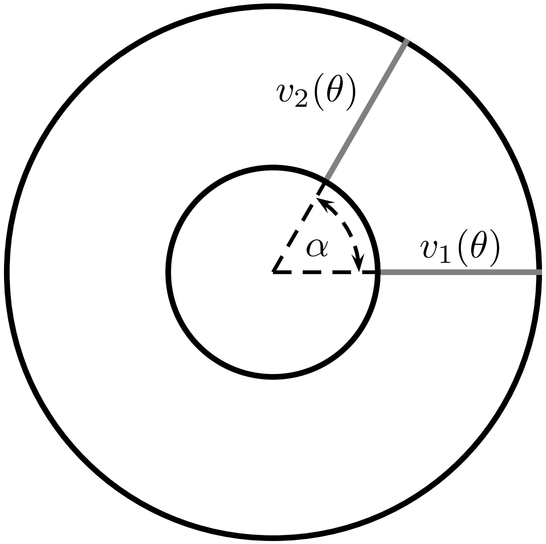


Brevik, Ellingsen, Milton, PRE **79** 041120 (2009); **80**, 021125 (2009); PRD **81**, 065031 (2010).

V. Annular Piston

- “Scalar Casimir Energies for Separable Coordinate Systems: Application to Semi-transparent Planes in an Annulus,” Wagner, Milton, K. Kirsten, arXiv:0912.2374, to appear in the proceedings of QFEXT09
- “Casimir Effect for a Semitransparent Wedge and an Annular Piston,” Milton, Wagner, Klaus Kirsten, arXiv:0911.1123, Phys. Rev. D **80**, 125028 (2009)

Annular piston–ST plates



We use multiple scattering in the angular coordinates, and an eigenvalue condition in the radial coordinates—equally well solvable with radial Green's functions, but generalizable.

Quantum vacuum energy

We start from the formula for the Casimir energy in terms of the Green's function,

$$E = \frac{1}{2i}\int \frac{d\omega}{2\pi} 2\omega^2 \mathrm{Tr}\,(\mathcal{G} - \mathcal{G}^{(0)}),$$

where the trace denotes the integration over spatial coordinates, and we have again subtracted out the vacuum contribution. The Green's function $\mathcal{G}(\mathbf{r}, \mathbf{r}')$ will satisfy the equation

$$\left[-\nabla^2 - \omega^2 + V(\mathbf{r})\right] \mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

while $\mathcal{G}^{(0)}$ has $V(\mathbf{r}) = 0$.

Cylindrical geometry of annulus

Boundary conditions: $G = 0$ at $\rho = a$ and $\rho = b$.
Potential: $V(\mathbf{r}) = v(\theta)/\rho^2$.
Green's function ($\eta =$ separation constant):

$$\mathcal{G}(\mathbf{r}, \mathbf{r}'; \omega) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} \sum_{\eta} R_\eta(\rho; \omega, k) R_\eta(\rho'; \omega, k)$$
$$\times g_\eta(\theta, \theta').$$

Eigenfunction and Green's function

Normalized radial eigenfunctions:

$$\left[-\rho\frac{d}{d\rho}\rho\frac{d}{d\rho} - (\omega^2 - k^2)\rho^2\right] R_\eta(\rho;\omega,k) = \eta^2 R_\eta(\rho;\omega,k),$$

BC: $R_\eta(a;\omega,k) = R_\eta(b;\omega,k) = 0$. Reduced Green's function:

$$\left[-\frac{d^2}{d\theta^2} + \eta^2 + v(\theta)\right] g_\eta(\theta,\theta') = \delta(\theta-\theta'),$$

with periodic boundary conditions.

Vacuum energy

$$E = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}2\omega^2\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int dz\sum_\eta\int_a^b \rho\,d\rho\,R_\eta^2(\rho;\omega,k)$$

$$\times\int d\theta\left[g_\eta(\theta,\theta)-g_\eta^{(0)}(\theta,\theta)\right].$$

or ($\omega\to i\zeta$, $\zeta^2+k_z^2=\kappa^2$),

$$\mathcal{E} = -\frac{1}{4\pi}\int_0^{\infty}\kappa^3 d\kappa\sum_\eta\int_a^b \rho\,d\rho\,R_\eta^2(\rho;\kappa)$$

$$\times\int d\theta\left[g_\eta(\theta,\theta)-g_\eta^{(0)}(\theta,\theta)\right].$$

Radial Integral

Using the differential equation, the desired radial integral is found to be concisely written as

$$\int_a^b \rho \, d\rho \, R_\eta^2(\rho;\kappa) = -\frac{\eta}{\kappa} \frac{\frac{\partial}{\partial\kappa}\tilde{R}_\eta(b;\kappa)}{\frac{\partial}{\partial\eta}\tilde{R}_\eta(b;\kappa)},$$

where $\tilde{R}_\eta(r;\kappa)$ is a solution of the differential equation which satisfies $\tilde{R}_\eta(a;\kappa) = 0$. Here the normalization is

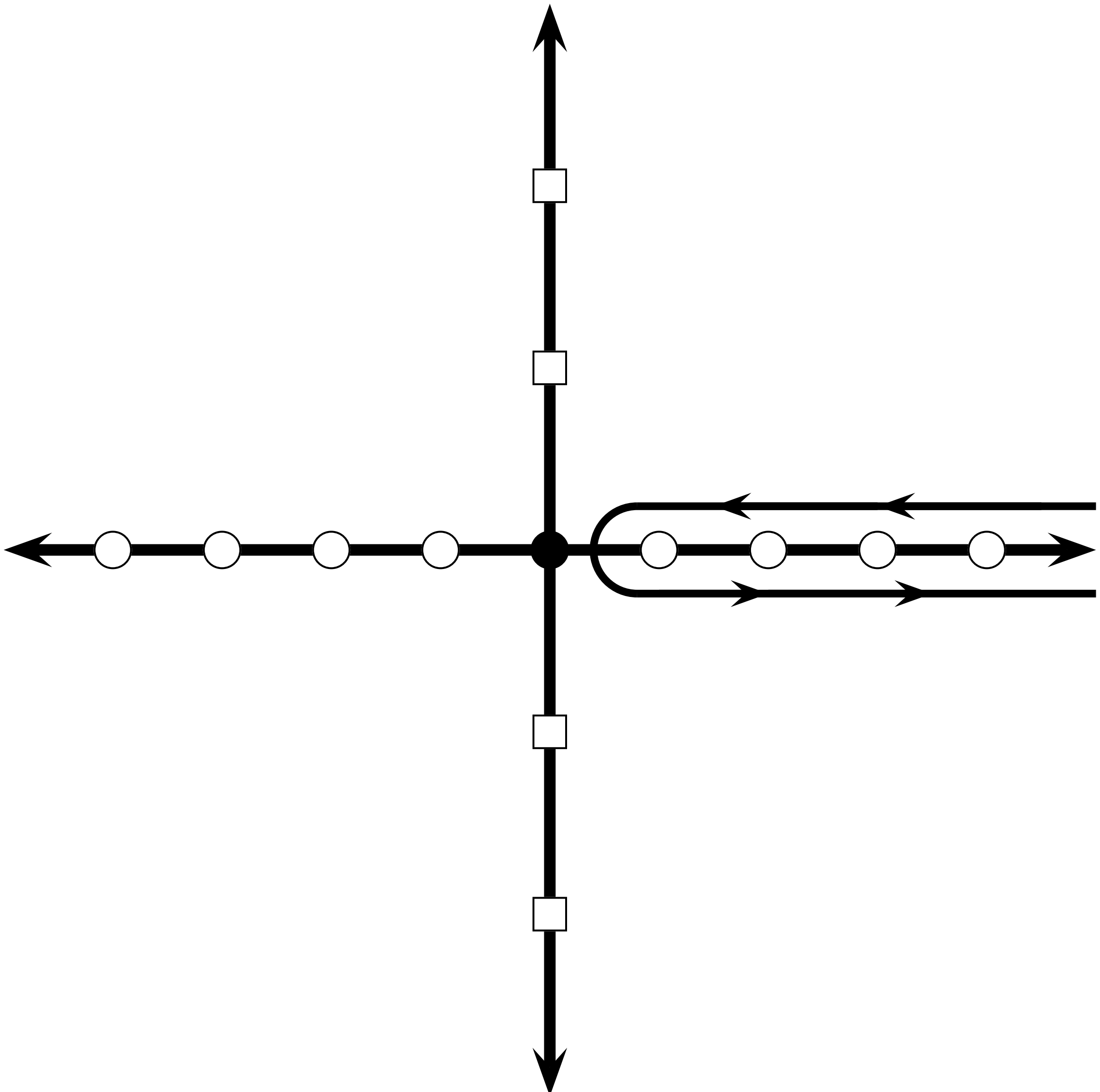
$$\int_a^b \frac{d\rho}{\rho} R_\eta^2(\rho;\kappa) = 1.$$

Argument principle

The eigenvalues are given by the zeros of $D(\eta) = R_\eta(b;\kappa)$. So we have

$$\sum_\eta \int_a^b \rho\, d\rho R_\eta^2(\rho;\kappa) = \frac{1}{2\pi i}\int_\gamma d\eta \frac{\frac{\partial}{\partial\eta}\tilde{R}_\eta(b;\kappa)}{\tilde{R}_\eta(b;\kappa)} \times \left(-\frac{\eta}{\kappa}\frac{\frac{\partial}{\partial\kappa}\tilde{R}_\eta(b;\kappa)}{\frac{\partial}{\partial\eta}\tilde{R}_\eta(b;\kappa)} \right) = -\frac{1}{2\pi i}\int_\gamma d\eta \frac{\eta}{\kappa}\frac{\partial}{\partial\kappa}\ln \tilde{R}_\eta(b;\kappa).$$

Contour of integration



The Radial Solutions

We need the solution of the modified Bessel differential equation, of imaginary order, which is zero for $\rho = a$ for all values of η and κ. An obvious solution is

$$\tilde{R}_\eta(\rho;\kappa) = K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa\rho) - \tilde{I}_{i\eta}(\kappa a)K_{i\eta}(\kappa\rho) = \tilde{R}_{-\eta}(\rho,\kappa)$$

where

$$\tilde{I}_\nu = \frac{1}{2}(I_\nu + I_{-\nu}) = \frac{\sin\nu\pi}{i\pi}L_\nu.$$

Reduced Green's Function

Free reduced Green's function:

$$g_\eta^{(0)}(\theta,\theta') = \frac{1}{2\eta}\left(-\sinh\eta|\theta-\theta'| + \frac{\cosh\eta\pi}{\sinh\eta\pi}\cosh\eta|\theta-\theta'$$

For a single potential $v(\theta)=\lambda\delta(\theta-\alpha)$ for $\theta,\theta'\in[\alpha,2\pi+\alpha]$: $g_\eta(\theta,\theta') =$

$$= \frac{1}{2\eta}\left(-\sinh\eta|\theta-\theta'| + \frac{2\eta\cosh\eta\pi\cosh\eta|\theta-\theta'|}{2\eta\sinh\eta\pi+\lambda\cosh\eta\pi}\right.$$

$$-\lambda\frac{\cosh\eta(2\pi+2\alpha-\theta-\theta')-\cosh 2\eta\pi\cosh\eta|\theta-\theta'|}{[2\eta\sinh\eta\pi+\lambda\cosh\eta\pi]2\sinh\eta\pi}$$

Trace of Green's fn

The quantity of interest, tr $\left(g - g^{(0)}\right)$, is then

$$\int_\alpha^{2\pi+\alpha} d\theta \left[g_\eta(\theta,\theta) - g_\eta^{(0)}(\theta,\theta) \right]$$

$$= \frac{-\lambda(\sinh\eta\pi\cosh\eta\pi + \eta\pi)}{2\eta^2\sinh\eta\pi(2\eta\sinh\eta\pi + \lambda\cosh\eta\pi)}$$

$$= \frac{1}{2\eta}\frac{\partial}{\partial\eta}\ln\left(1 + \frac{\lambda}{2\eta}\coth\eta\pi\right)$$

$$= \frac{1}{2\eta}\frac{\partial}{\partial\eta}\ln\left(1 + \lambda g_\eta^{(0)}(\alpha,\alpha)\right),$$

precisely of the expected form.

Casimir energy for single plane

The final form for the Casimir energy for a single radial δ-function potential in the annular region is

$$\mathcal{E} = \frac{1}{16\pi^2 i}\int_0^\infty \kappa^2 d\kappa \int_\gamma d\eta \left(\frac{\partial}{\partial\kappa}\ln\left[K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa b) - \tilde{I}_{i\eta}(\kappa a)K_{i\eta}(\kappa b)\right]\right)\left(\frac{\partial}{\partial\eta}\ln\left[1+\frac{\lambda}{2\eta}\coth\eta\pi\right]\right).$$

MS derivation

This result may also be obtained by the multiple scattering formalism, which says that

$$E = -\frac{1}{2i}\mathrm{Tr}\,\ln GG^{(0)-1} = \frac{1}{2i}\mathrm{Tr}\,\ln(1+G^{(0)}V).$$

The above agrees if IBP:

$$\mathcal{E} = \frac{1}{8\pi^2 i}\int_0^\infty d\kappa\,\kappa\int_\gamma d\eta\left(\frac{\partial}{\partial\eta}\ln\left[K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa b)\right.\right.$$
$$\left.\left.-\,\tilde{I}_{i\eta}(\kappa a)K_{i\eta}(\kappa b)\right]\right)\ln\left[1+\lambda g^{(0)}(\alpha,\alpha)\right].$$

Two Semitransparent Planes

If we want to look at an explicitly finite quantity we will need to look at the interaction energy between two semitransparent planes.
Multiple-scattering formalism:

$$E = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathrm{Tr}\,\ln(1-\mathcal{G}^{(1)}V_1\mathcal{G}^{(2)}V_2).$$

The subscripts on the Vs represent the potentials $V_1(\mathbf{r}) = \lambda_1\delta(\theta)/\rho^2$, and $V_2(\mathbf{r}) = \lambda_2\delta(\theta-\alpha)/\rho^2$. The Green's functions with superscript (i) represent the interaction with only a single potential V_i.

Simplified interaction energy

$$\mathcal{E} = \frac{1}{4\pi}\int_0^\infty \kappa\, d\kappa \sum_\eta \ln\left(1 - \mathrm{tr}\, g_\eta^{(1)} v_1 g_\eta^{(2)} v_2\right),$$

$g_\eta^{(i)}$ given above. Then

$$\mathrm{tr}\, g_\eta^{(1)} v_1 g_\eta^{(2)} v_2$$

$$= \frac{\lambda_1\lambda_2 \cosh^2\eta(\pi-\alpha)}{(2\eta\sinh\eta\pi + \lambda_1\cosh\eta\pi)\,(2\eta\sinh\eta\pi + \lambda_2\cosh\eta\pi)}.$$

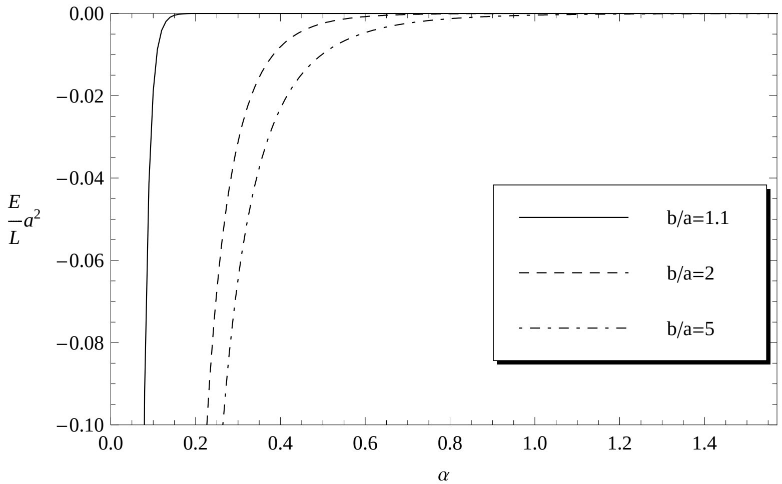
Energy for annular Casimir piston

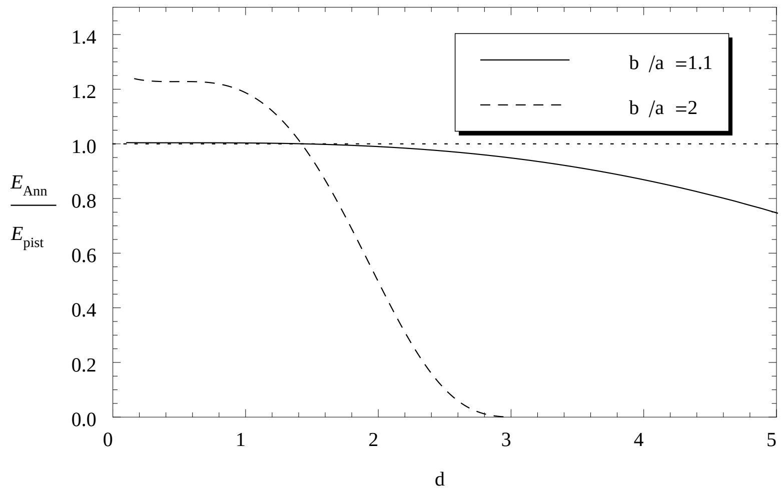
Using the argument principle to determine the angular eigenvalues, we get the following expression for the energy for an annular Casimir piston: ($\tilde{I}_\nu = \frac{1}{2}(I_\nu + I_{-\nu})$) $\mathcal{E} =$

$$\int_0^\infty \frac{\kappa d\kappa}{8\pi^2 i} \int_\gamma d\eta \frac{\partial}{\partial\eta} \ln\left[K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa b) - \tilde{I}_{i\eta}(\kappa a) K_{i\eta}(\kappa b)\right]$$

$$\times \ln\left(1 - \frac{\lambda_1\lambda_2 \cosh^2\eta(\pi-\alpha)/\cosh^2\eta\pi}{(2\eta\tanh\eta\pi + \lambda_1)(2\eta\tanh\eta\pi+\lambda_2)}\right).$$

MS calculation for annular piston





Energy/length for annular piston as function of angle (left), and compared to energy/length for rectangular piston (right). Here $d = \frac{b+a}{2} \sin \frac{\alpha}{2}$, and plateaus may be understood from the PFA,

$$\frac{\mathcal{E}_{\text{PFA}}}{\mathcal{E}_{\|}} = \frac{1}{16} \frac{b^2}{a^2} \left(1 + \frac{a}{b}\right)^4 .$$

VI. Triangular cylinder

For an equilateral triangle of height h, the scalar eigenmodes corresponding to Dirichlet boundary conditions are known explicitly

$$\gamma_l^2 = \frac{2}{3}\left(\frac{\pi}{h}\right)^2 (l_1^2 + l_2^2 + l_3^2),$$

where

$$l_1 + l_2 + l_3 = 0, \quad l_i \neq 0.$$

[Preliminary. Work in progress with Elom Abalo.]

Casimir energy

In d transverse dimensions, the Casimir energy is

$$\mathcal{E} = -\frac{\Gamma(-1/2 - d/2)}{2^{2+d}\pi^{(d+1)/2}}\sum_l (\gamma_l^2)^{(d+1)/2}$$

which can be analytically continued and summed by means of the Chowla-Selberg formula (which we used to find the temperature dependence for the diaphanous wedge), which is exceedingly rapidly convergent.

$$\mathcal{E} = +\frac{0.0177891}{h^2}.$$

Chowla-Selberg (Kronecker):

$$\sum_{m,j\in\mathbb{Z}}{}'' (am^2+bmj+cj^2)^{-q} = 2\zeta(2q)a^{-q}$$

$$+\frac{2^{2q}\sqrt{\pi}a^{q-1}\Gamma(q-\frac{1}{2})\zeta(2q-1)}{\Gamma(q)\Delta^{q-\frac{1}{2}}}$$

$$+\frac{2^{q+\frac{5}{2}}\pi^q}{\Gamma(q)\Delta^{\frac{1}{2}(q-\frac{1}{2})}\sqrt{a}}\sum_{l=1}^{\infty} l^{q-\frac{1}{2}}\sigma_{1-2q}(l)$$

$$\times\cos(l\pi b/a)K_{q-\frac{1}{2}}(\pi l\sqrt{\Delta}/a),$$

$\Delta = 4ac-b^2$, $\sigma_w(l) = \sum_{\nu|l}\nu^w$, where ν are summed over the divisors of l.

Poisson sum formula

We can also evaluate the eigenvalue sum by use of the Poisson sum formula,

$$\sum_{l=-\infty}^{\infty} f(l) = 2\pi \sum_{k=-\infty}^{\infty} \tilde{f}(k),$$

in terms of the Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{2\pi i k \alpha} f(\alpha).$$

Alternative evaluation

The unregulated energy for the equilateral triangle is

$$\mathcal{E} = -\frac{1}{8\pi}\frac{1}{6}\int_0^\infty \frac{dt}{t}\frac{1}{t}\left\{\sum_{l_1,l_2=-\infty}^{\infty} e^{-\beta t(l_1^2+l_2^2+l_1 l_2)} -3\sum_{l=-\infty}^{\infty} e^{-\beta t l^2} + 2\right\},$$

$\beta = \frac{4}{3}(\pi/h)^2$, where the double sum may be broken up using

$$l_1^2 + l_2^2 + l_1 l_2 = 3m^2 + n^2, \quad l_{1,2} = m \pm n.$$

Two equivalent forms

$$\mathcal{E} = -\frac{1}{144\pi^2 h^2}\Bigg[3\sqrt{3}\zeta(4) - 8\pi\zeta(3)$$

$$+ 4\pi^2(12)^{3/4}\sum_{l=1}^{\infty}(-1)^l l^{-3/2}\sigma_3(l)K_{3/2}(\sqrt{3}\pi l)\Bigg]$$

$$= -\frac{1}{144\pi^2 h^2}\Bigg[\frac{10}{\sqrt{3}}\zeta(4) - 12\pi\zeta(3)$$

$$+16\sqrt{3}\sum_{k,l=1}^{\infty}\frac{1+(-1)^{l+k}}{(k^2+l^2/3)^2}\Bigg] = 0.0177891/h^2.$$

2 terms ($\sim 10^5$ terms) gives 6-figure accuracy.

$\mathcal{E}_c$ for square waveguide

Same methods for evaluating the Casimir energy for a square waveguide (side a) (Lukosz/ Ambjørn and Wolfram):

$$\mathcal{E} = -\frac{1}{32\pi^2 a^2}\left[2\zeta(4) - \pi\zeta(3) + 8\pi^2 \sum_{l=1}^{\infty} l^{3/2}\sigma_3(l) K_{3/2}(\pi l)\right.$$

$$= -\frac{1}{32\pi^2 a^2}\left[4\zeta(4) - 2\pi\zeta(3) + 4\sum_{k,l=1}^{\infty}\frac{1}{(k^2+l^2)^2}\right]$$

$$= 0.00483155/a^2$$

The C-S formula is extraordinarily convergent.

Other triangles

By bifurcating the square, we can obtain the isosceles right triangle, and by bifurcating the equilateral triangle we can get the $30°$-$60°$-$90°$ triangle:

$$\mathcal{E}_{\rm iso} = \frac{1}{2}\mathcal{E}_{\rm sq} + \frac{\zeta(3)}{16\pi a^2} = \frac{0.0263299}{a^2},$$

$$\mathcal{E}_{369} = \frac{1}{2}\mathcal{E}_{\rm et} + \frac{\zeta(3)}{8\pi h^2} = \frac{0.0567229}{h^2},$$

To be compared to the result for a circle

$$\mathcal{E}_{\rm circ} = \frac{0.0006148}{a^2}.$$

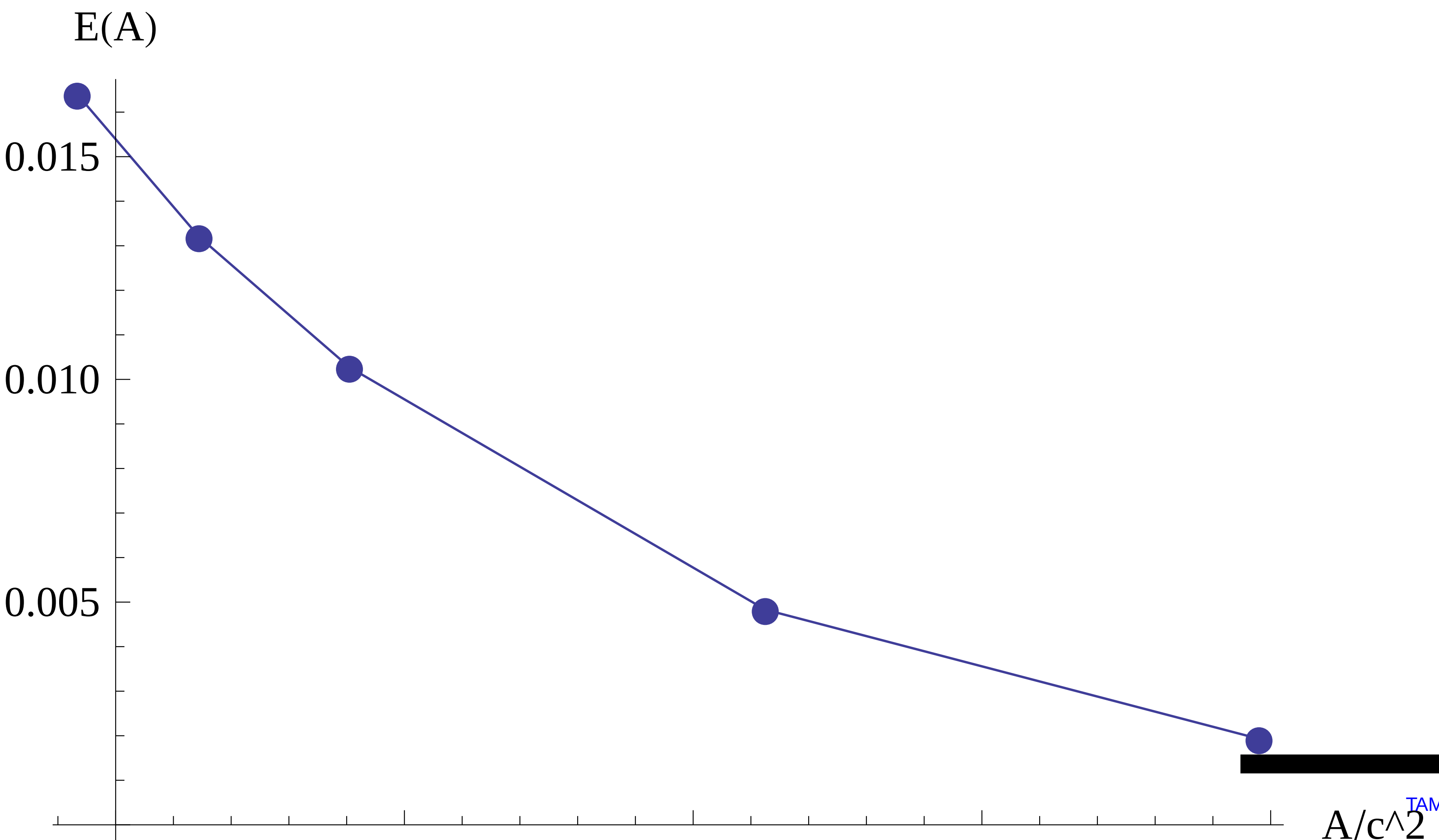
H modes

We can also get results for Neumann boundary conditions (H or TE modes)

$$\mathcal{E}^N_{\rm sq} = \mathcal{E}^D_{\rm sq} - \frac{\zeta(3)}{8\pi a^2} = -\frac{0.0429968}{a^2},$$
$$\mathcal{E}^N_{\rm et} = \mathcal{E}^D_{\rm et} - \frac{\zeta(3)}{6\pi h^2} = -\frac{0.045982}{h^2},$$
$$\mathcal{E}^N_{\rm iso} = \frac{1}{2}\mathcal{E}^N_{\rm sq} - \frac{\zeta(3)}{16\pi a^2} = -\frac{0.0454125}{a^2},$$
$$\mathcal{E}^N_{369} = \frac{1}{2}\mathcal{E}^N_{\rm et} - \frac{\zeta(3)}{8\pi h^2} = -\frac{0.0708193}{h^2}.$$

Systematic dependence of $\mathcal{E}_c^D$

$\mathcal{E}(a) \rightarrow \mathcal{E}(A)$, $A =$ cross-sectional area, as a function of A/c^2, $c =$ circumference of waveguide.



VII. Conclusions

- I believe finite Casimir self-energies can be well-defined, in that they are robustly calculable.
- Self-energies, however, are difficult to ascribe physical meaning to, except in the presence of gravity.
- Self-energies gravitate like all other forms of energy, consistent with the equivalence principle.
- Therefore, it is consistent to remove the divergent parts through some sort of renormalization.

Mutual energy unambiguous

- Casimir interaction energies between rigid bodies are finite and observable without question.
- Multiple-scattering techniques are very effective at yielding these energies, but they are equally applicable to calculating self-energies.