The problem

Zeta Functio 00 00000 Analytic Continuation

Casimir Energy and Force 0 000

Potentials with compact support

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Zeta Function 00 00000 Analytic Continuation

Casimir Energy and Force 0 000

Outline

 The problem The Idea The Statement The Assumptions 2 Zeta Function Secular Equation Characteristic Function Asymptotics for large y3 Analytic Continuation General Setting Analytic Continuation 4 Casimir Energy and Force Determinant and Energy Force

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The Idea			

Pistons with Compact Supported Potentials

Consider a piston



with Dirichlet boundary conditions and middle plate represented by V(x) BAYLO

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The problem ○ ○	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force 0 000	
The Idea				
Zeta Func	Zeta Function			

• State the problem as an eigenvalue equation



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The problem ○ ○	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force 0 000
The Idea			
Zeta Euro	tion		

- State the problem as an eigenvalue equation
- Write down the spectral zeta function



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The problem ○ ○	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force o ooo
The Idea			
Zeta Function			

- State the problem as an eigenvalue equation
- Write down the spectral zeta function
- Find the Casimir energy and functional determinant

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The problem ○ ○	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force o ooo
The Idea			
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The problem ○ ○	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force 0 000
The Idea			
Zota Euro	tion		

- State the problem as an eigenvalue equation
- Write down the spectral zeta function
- Find the Casimir energy and functional determinant
 - Find analytic continuation
 - Evaluate $\zeta'(0)$ and $\zeta\left(-\frac{1}{2}\right)$

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The problem	Zeta Function	Analytic Continuation	Casimir Energy and Force
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The Statement			

Eigenvalue Problem

We want to solve the eigenvalue equation

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)\right)\phi(x)=\lambda^2\phi(x)$$

in [0, L] with Dirichlet boundary conditions.

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The problem	Zeta Function	Analytic Continuation	Casimir Energy and Force
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O The Statement			

Eigenvalue Problem

We want to solve the eigenvalue equation

$$\left(-\frac{\partial^2}{\partial x^2}+V(x)\right)\phi(x)=\lambda^2\phi(x)$$

in [0, L] with Dirichlet boundary conditions. Let V(x) be supported in [a, a + w] such that w > 0 and

$$\int_{a}^{a+w} V(x)dx = \sigma$$

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is fixed.

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The problem ○○ ●	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force 0 000		
The Assumptions					
Assumptions on ϕ					

• Continuous



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The problem ○○ ●	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force o ooo		
The Assumptions					
Assumption	Assumptions on ϕ				

- Continuous
- Incommensurable lengths $(a/L \notin \mathbb{Q})$



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The problem ○○ ●	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force o ooo	
The Assumptions				
Assumptions on ϕ				

- Continuous
- Incommensurable lengths $(a/L \notin \mathbb{Q})$
- Normalization $(\phi(a) = \phi(a + w) = 1)$

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Secular Equation			
Secular Condi	ition		
Integrating t	he equation around	the support gives	

$$\lambda^{2} \int_{a-\epsilon}^{a+w+\epsilon} \phi(x) dx = - \int_{a-\epsilon}^{a+w+\epsilon} \frac{\partial^{2} \phi(x)}{\partial x^{2}} dx + \int_{a-\epsilon}^{a+w+\epsilon} V(x) \phi(x) dx$$

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Letting $\epsilon \rightarrow 0$ and using the solutions of the differential equation, lead to

$$\lambda \cot(\lambda a) + \lambda \cot(\lambda(L - w - a)) + \sigma - \lambda^{2} w$$
$$= \int_{a}^{a+w} \phi'(x) \left[\int_{a}^{x} (V(u) - \lambda^{2}) du \right] dx$$

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Characteristic Function			

Characteristic Function

In order to remove singularities at the origin, define

$$F(\lambda) = -\frac{1}{\lambda}\sin(\lambda(L-w)) - \frac{\sigma}{\lambda^2}\sin(\lambda a)\sin(\lambda(L-w-a)) +w\sin(\lambda a)\sin(\lambda(L-w-a)) +\frac{\sin(\lambda a)\sin(\lambda(L-w-a))}{\lambda^2} \int_{a}^{a+w} \phi'(x) \left[\int_{a}^{x} (V(u) - \lambda^2) du\right] dx$$

 $= \underset{U=N-I}{\overset{N}{\longrightarrow}} \underset{V=E}{\overset{Y}{\rightarrow}} \underset{R=S-I}{\overset{L}{\longrightarrow}} \underset{T=Y}{\overset{N}{\rightarrow}} \underset{T=Y}{\overset{N}{\overset{N}{\rightarrow}} \underset{T=Y}{\overset{N}{\rightarrow}} \underset{T=Y}{\overset{N}{\rightarrow}} \underset{T=Y}{\overset{N}{\rightarrow}} \underset{T=Y}{\overset{N}{\overset{N}{}} \underset{T=Y}{\overset{N}{}$

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Characteristic Function					
Asymptotics	for large <i>y</i>				

In order to write the zeta function as a contour integral and deform it to the imaginary axis, we set $\lambda = iy$, and when $y \to \infty$,

$$\frac{\partial^2 \phi(x)}{\partial x^2} + V(x)\phi(x) + y\phi(x) \sim \frac{\partial^2 \phi(x)}{\partial x^2} + y\phi(x)$$

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Characteristic Function			

so that

$$\int\limits_{a}^{a+w} \phi(x) dx \sim c_V$$

where c_V is a constant depending on V(x) and independent of a

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The problem	Zeta Function	Analytic Continuation	Casimir Energy and Force
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Characteristic Function			

Thus,

$$F(iy) \sim \left(w - \left(rac{1}{2y} + rac{d_V}{4y^2}
ight)
ight) e^{y(L-w)}$$

where

$$d_V = \sigma + c_V$$

depends only on the potential V(x) and is independent of a.

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Therefore, we have the asymptotics of the logarithm

$$\ln F(iy) \sim y(L-w) + \ln w - \sum_{k=1}^{\infty} y^{-k} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} {\binom{k-j}{j}} 2^{-k} w^{j-k} d_V^j$$

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Therefore, we have the asymptotics of the logarithm

$$\ln F(iy) \sim y(L-w) + \ln w - \sum_{k=1}^{\infty} y^{-k} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-j}{j} 2^{-k} w^{j-k} d_V^j$$

Remark

In the limit when w = 0 the asymptotic expansion contains a logarithmic term in y recovering the semitransparent result

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The problem 00 0 0	Zeta Function 00 00000	Analytic Continuation	Casimir Energy and Force 0 000
General Setting			
$[0,L] imes \mathcal{N}$ se	etting		

Considering the problem in $[0, L] \times N$, where N is a (D-1)-dimensional compact Riemannian manifold possibly with boundary, we have that

$$\zeta(s) = \sum_{\lambda,\ell} \left(\lambda^2 + \eta_\ell^2
ight)^{-s} \quad ext{ for } \Re(s) > rac{D}{2}$$

where η_ℓ is the spectrum in ${\mathcal N}$

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Analytic Continuation			

Therefore



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The problem	Zeta Function	Analytic Continuation	Casimir Energy and Force
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Analytic Continuation			

where

$$d_0 = d_V$$
 and $d_k = \sum_{j=0}^{\left\lfloor rac{k}{2}
ight
ceil} {k-j \choose j} 2^{-k} w^{j-k} d_V^j$

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 Determinant and Energy 			

Functional Determinant and Casimir Energy

Letting N = D - 1 we can calculate $\zeta'(0)$ which gives the functional determinant

 $\exp(\zeta'(0))$



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The problem 00 0 0	Zeta Function 00 00000	Analytic Continuation 0 00	Casimir Energy and Force
Determinant and Energy			

Functional Determinant and Casimir Energy

Letting N = D - 1 we can calculate $\zeta'(0)$ which gives the functional determinant

$\exp\left(\zeta'(0) ight)$

and setting N = D leads to the Casimir energy

$$\zeta\left(-\frac{1}{2}\right)$$

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Force			
Casimir Fo	orce		

In order to calculate the Casimir Force, we can compute

$$-\frac{1}{2}\frac{\partial}{\partial a}\zeta\left(-\frac{1}{2}\right)$$



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Force			

Attractive or repulsive?

• For some specific potentials, the force is attractive to the closest wall (delta, rectangular)



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Force			

Attractive or repulsive?

- For some specific potentials, the force is attractive to the closest wall (delta, rectangular)
- Attractive for all potentials? (It seems to be the case)

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Zeta Function

Analytic Continuation

Casimir Energy and Force $\circ \\ \circ \circ \bullet$



Thank You!

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