Leading order contribution to lateral Casimir force between corrugated dielectric slabs.

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Outline

Lateral Casimir force

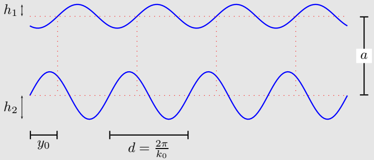
Consider potential

$$V_i(z, y) = \lambda_i \, \delta(z - a_i - h_i(y)), \quad i = 1, 2, \quad a = a_2 - a_1 > 0,$$

where, the functions $h_i(y)$ describe the corrugations on the plate.

Casimir energy for a configuration when one of the plate is laterally shifted by an amount y_0,

$$h_1(y + y_0), \quad h_2(y),$$



can be written in the form

$$\Delta E(a, h_i, y_0) = E - E^{(0)}(a) = E_1(a, h_1) + E_2(a, h_2) + E_{12}(a, h_i, y_0),$$

where E_{12} isolates the interaction energy due to the lateral shift.

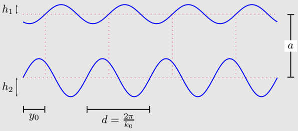
Lateral Casimir force is defined as the negative change in energy due to the lateral shift:

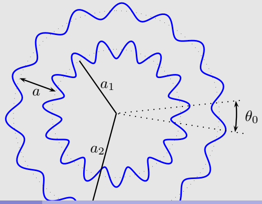
$$F_{\text{Lat}}(a, h_i, y_0) = -\frac{\partial}{\partial y_0} \Delta E = -\frac{\partial E_{12}}{\partial y_0}.$$

Motivation

1997-2010: Starting with first theoretical study by Golestanian et. al. in 1997, various theoretical and experimental analysis of lateral Casimir force has been done till now. One remarkable theoretical calculation for lateral Casimir force between dielectric gratings was done exactly by Lambrecht and Marachevsky, which shows 0.1% accuracy between theory and experiment.

In noncontact Gears I and II (PRD 78, 065018, 065019) we presented next-to-leading order contibution to the lateral Casimir force in planar geometry and leading order contribution in cylinderical geometry due to scalar field interacting with semi-transparent delta potential.



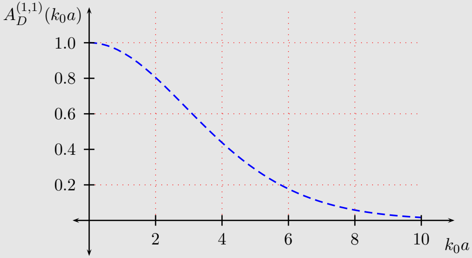


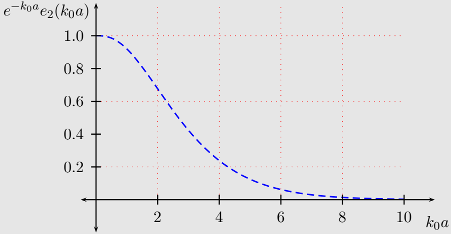
Leading order perturbation-Dirichlet and Weak

Leading order contribution to the lateral Casimir force in Dirichlet limit and weak limit for sinusoidal corrugations are

$$F^{(2)}_{\text{Lat,D}} = 2k_0 a \sin(k_0 y_0) \left|F^{(0)}_{\text{Cas,D}}\right| \frac{h_1}{a}\frac{h_2}{a} A^{(1,1)}_D(k_0 a),$$

$$F^{(2)}_{\text{Lat,W}} = k_0 a \sin(k_0 y_0) \left|F^{(0)}_{\text{Cas,W}}\right| \frac{h_1}{a}\frac{h_2}{a} A^{(2)}_W(k_0 a).$$

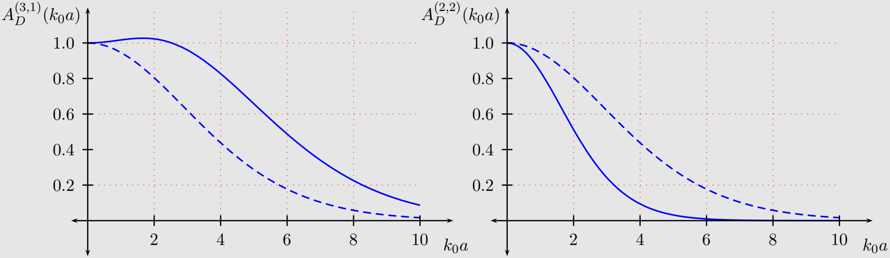




Next-to-leading order perturbation - Dirichlet

Next-to-Leading order contribution to the lateral Casimir force in the Dirichlet limit for sinusoidal corrugations is

$$F^{(4)}_{\text{Lat,D}} = 2\,k_0 a \sin(k_0 y_0) \left| F^{(0)}_{\text{Cas,D}} \right| \frac{h_1}{a}\frac{h_2}{a}\,\frac{15}{4} \times \left[\left(\frac{h_1^2}{a^2} + \frac{h_2^2}{a^2} \right) A_D^{(3,1)}(k_0 a) - 2\cos(k_0 y_0)\,\frac{h_1}{a}\frac{h_2}{a}\,A_D^{(2,2)}(k_0 a) \right].$$



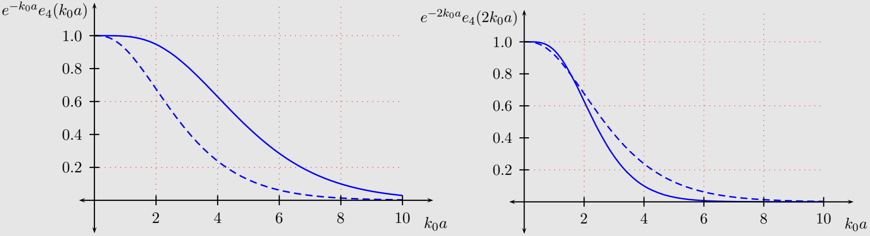
Next-to-leading order perturbation - Weak

The next correction to the lateral Casimir force for the the weak coupling case is

$$F^{(4)}_{\text{Lat,W}} = k_0 a \sin(k_0 y_0) \left| F^{(0)}_{\text{Cas,W}} \right| \frac{h_1}{a} \frac{h_2}{a} \frac{3}{2} \times \left[\left(\frac{h_1^2}{a^2} + \frac{h_2^2}{a^2} \right) A^{(4)}_W(k_0 a) - 2\cos(k_0 y_0) \frac{h_1}{a} \frac{h_2}{a} A^{(4)}_W(2k_0 a) \right],$$

where

$$A^{(n)}_W(t_0) = e^{-t_0} \sum_{m=0}^{n} \frac{t_0^m}{m!} = \frac{e_n(t_0)}{e^{t_0}}.$$

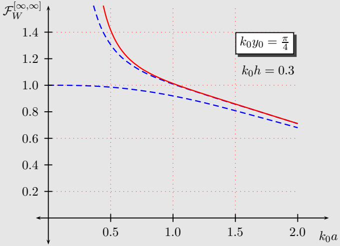


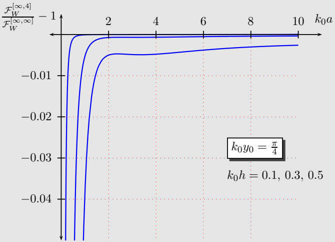
Analysis

Weak coupling limit - Non-perturbative results

$$\mathcal{F}_W^{[\infty,\infty]} = -\frac{(k_0 a)^3}{\sin(k_0 y_0)} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t} \,\text{Re}\left[\frac{\sin(t + k_0 y_0)}{[(t + i k_0 a)^2 + \{k_0 r(t)\}^2]^{\frac{3}{2}}} \right],$$

for $2h < a$.





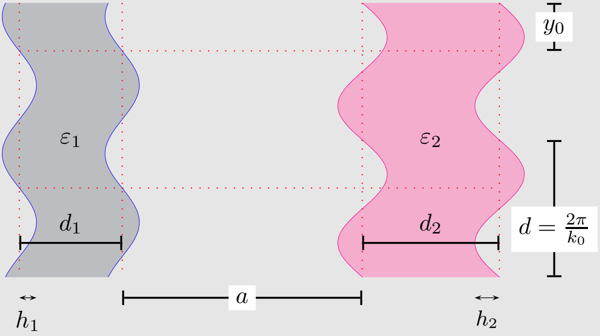
We observe that the perturbative results, when the next-to-leading order is included, compares with the exact result remarkably well for $k_0 h \ll 1$ and $2h < a$. Similar results should hold for the Dirichlet limit also.

Electromagnetic case: Statement of the problem

We consider two dielectric slabs

$$V_i(z, y; \omega) = \omega^2(\epsilon_i(\omega) - 1)\,[\theta(z - a_i - h_i(y)) - \theta(z - b_i - h_i(y))]\,,$$

where $i = 1, 2$



where, the thickness of the individual slabs is $d_i = b_i - a_i$, such that $a = a_2 - b_1 > 0$ represents the distance between the slabs.

Lateral Casimir force

The vacuum energy arising from the fluctuation of the electromagnetic field in presence of the potential is given by

$$E = -\frac{i}{2} \int \frac{d\omega}{2\pi} \operatorname{Tr} \ln \mathbf{\Gamma} \, \mathbf{\Gamma}_0^{-1},$$

where $\mathbf{\Gamma}$ is the Green's dyadic which obeys

$$\left[-\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times + \omega^2 \mathbf{1} + \mathbf{V}_1(\mathbf{x}) + \mathbf{V}_2(\mathbf{x}) \right] \mathbf{\Gamma}(\mathbf{x}, \mathbf{x}'; \omega) = -\omega^2 \mathbf{1} \, \delta(\mathbf{x} - \mathbf{x}'),$$

and $\mathbf{\Gamma}_0$ is the Green's dyadic for the free space.

As shown earlier Casimir energy for a configuration when one of the slab is laterally shifted by an amount y_0 is

$$\Delta E(a, h_i, y_0) = E - E^{(0)}(a) = E_1(a, h_1) + E_2(a, h_2) + E_{12}(a, h_i, y_0).$$

Lateral Casimir force is defined as the negative change in energy due to the lateral shift

$$F_{\text{Lat}}(a, h_i, y_0) = -\frac{\partial}{\partial y_0} \Delta E = -\frac{\partial E_{12}}{\partial y_0}.$$

Formalism(contd.)

Interaction energy E_{12} is given by

$$E_{12} = \frac{i}{2} \int \frac{d\omega}{2\pi} \operatorname{Tr} \ln \left(1 - \boldsymbol{\Gamma}_1 \, \Delta \overline{V}_1 \, \boldsymbol{\Gamma}_2 \, \Delta \overline{V}_2\right).$$

where $\boldsymbol{\Gamma}_i$ is the Green's dyadic when one slab has corrugations.

$$\boldsymbol{\Gamma}_i = \left[\mathbf{1} - \boldsymbol{\Gamma}^{(0)} \, \Delta \overline{V}_i\right]^{-1} \boldsymbol{\Gamma}^{(0)},$$

where $\Delta \overline{V}_i = \overline{V}_i - \overline{V}_i^{(0)}$ is the deviation from the background potential $\overline{V}_i^{(0)}$ given by

$$\overline{V}_i^{(0)}(z;\omega) = (\epsilon_i(\omega) - 1) \, [\theta(z - a_i) - \theta(z - b_i)].$$

$\boldsymbol{\Gamma}^{(0)}$ is the Green's dyadic for no corrugation configuration.

Leading order contribution - perturbative expansion

Formally expanding the log and keeping terms to second order in corrugation amplitude (which requires expanding potential for small corrugation amplitude) we get

$$E_{12}^{(2)} = -\frac{1}{2} \int \frac{d\zeta}{2\pi} \, \mathrm{Tr} \ln \mathbf{\Gamma}^{(0)} \, \Delta \overline{V}_1^{(1)} \, \mathbf{\Gamma}^{(0)} \, \Delta \overline{V}_2^{(1)} \, ,$$

where

$$\Delta \, \overline{V}_i^{(1)}(z, y; i\zeta) = -h_i(y) \, (\epsilon_i(i\zeta) - 1) \, [\delta(z - a_i) - \delta(z - b_i)] \, ,$$

is the first term in expansion of potential for small corrugation amplitude. $\mathbf{\Gamma}^{(0)}$ is translationally invariant in x and y direction. So we can Fourier transform x and y directions

$$\frac{E_{12}^{(2)}}{L_x} = \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y'}{2\pi} \, \tilde{h}_1(k_y - k_y') \, \tilde{h}_2(k_y' - k_y) \, L^{(2)}(k_y, k_y'),$$

where L_x is the length in x direction and $\tilde{h}_i(k_y)$ are the Fourier transforms of the corrugation amplitude functions $h_i(y)$.

Leading order contribution(cont.)

Kernel $L^{(2)}(k_y, k_y')$ is given by

$$L^{(2)}(k_y, k_y') = -\frac{1}{2}\int \frac{d\zeta}{2\pi}\int \frac{dk_x}{2\pi} I^{(2)}(k_x, \zeta, k_y, k_y'),$$

where,

$$\begin{aligned} I^{(2)}(k_x, \zeta, k_y, k_y') = \; & (\epsilon_1(i\zeta) - 1)(\epsilon_2(i\zeta) - 1) \times \\ & \Big[\tilde{\boldsymbol{\Gamma}}^{(0)}_{ij}(a_2, a_1; k_x, k_y, \omega)\,\tilde{\boldsymbol{\Gamma}}^{(0)}_{ji}(a_1, a_2; k_x, k_y', \omega) \\ & -\tilde{\boldsymbol{\Gamma}}^{(0)}_{ij}(b_2, a_1; k_x, k_y, \omega)\,\tilde{\boldsymbol{\Gamma}}^{(0)}_{ji}(a_1, b_2; k_x, k_y', \omega) \\ & -\tilde{\boldsymbol{\Gamma}}^{(0)}_{ij}(a_2, b_1; k_x, k_y, \omega)\,\tilde{\boldsymbol{\Gamma}}^{(0)}_{ji}(b_1, a_2; k_x, k_y', \omega) \\ & +\tilde{\boldsymbol{\Gamma}}^{(0)}_{ij}(b_2, b_1; k_x, k_y, \omega)\,\tilde{\boldsymbol{\Gamma}}^{(0)}_{ji}(b_1, b_2; k_x, k_y', \omega)\Big], \end{aligned}$$

So the quantity we need to evaluate is the Green's dyadic for the background potential, which is the slabs without corrugation.

Evaluation of the reduced Green's dyadic

The Green's dyadic $\mathbf{\Gamma}^{(0)}$ obeys

$$\left[-\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times + \omega^2 \mathbf{1} + \omega^2 \mathbf{V}_1^{(0)}(\mathbf{x}) + \omega^2 \mathbf{V}_2^{(0)}(\mathbf{x})\right] \mathbf{\Gamma}^{(0)}(\mathbf{x}, \mathbf{x}'; \omega) = -\omega^2 \mathbf{1}\, \delta(\mathbf{x} - \mathbf{x}').$$

For planar geometry the problem reduces to solving for two scalar Green's function $g^E(z, z')$ and $g^H(z, z')$ which satisfies

$$\left[-\frac{\partial^2}{\partial z^2} + k^2 + \zeta^2 \epsilon(z)\right] g^E(z, z') = \delta(z - z'),$$

$$\left[-\frac{\partial}{\partial z} \frac{1}{\epsilon(z)} \frac{\partial}{\partial z} + \frac{k^2}{\epsilon(z)} + \zeta^2\right] g^H(z, z') = \delta(z - z'),$$

where $\epsilon(z) = 1 + V(z)$ and $k = k_x$ since we can choose $k_y = 0$.

Evaluation of the reduced Green's dyadic(cont.)

The reduced Green's dyadic in terms of $g^E(z,z')$ and $g^H(z,z')$ is

$$\mathbf{\Gamma}^{(0)}(z,z';k,0,\zeta) = \begin{bmatrix} \frac{1}{\epsilon(z')}\frac{\partial}{\partial z'}\frac{1}{\epsilon(z)}\frac{\partial}{\partial z}\, g^H(z,z') & 0 & \frac{i\,k}{\epsilon(z)\,\epsilon(z')}\frac{\partial}{\partial z}\, g^H(z,z') \\ 0 & -\zeta^2\, g^E(z,z') & 0 \\ \frac{i\,k}{\epsilon(z)\,\epsilon(z')}\frac{\partial}{\partial z}\, g^H(z,z') & 0 & \frac{k^2}{\epsilon(z)\,\epsilon(z')}\, g^H(z,z') \end{bmatrix}.$$

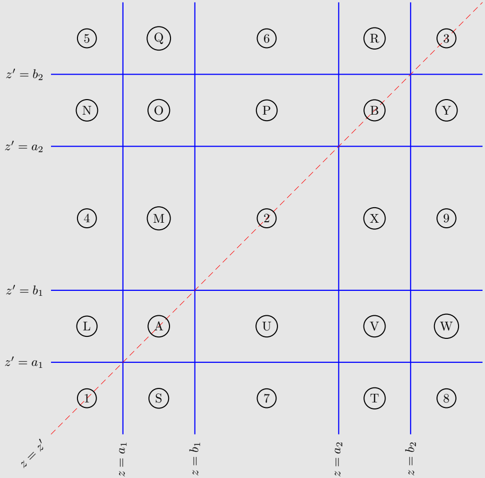
where we have dropped the delta functions, which do not contribute.

Evaluation of the reduced Green's dyadic(cont.)

Denoting $\mathbf{\Gamma}$ matrix components as Γ_{ij}, we can write the general form as

$$\mathbf{\Gamma}^{(0)}(z,z';k_x,k_y,\zeta) = \begin{bmatrix} \frac{k_x^2}{k^2}\Gamma_{11} + \frac{k_y^2}{k^2}\Gamma_{22} & \frac{k_x k_y}{k^2}\Gamma_{11} - \frac{k_x k_y}{k^2}\Gamma_{22} & \frac{k_x}{k}\Gamma_{13} \\ \frac{k_x k_y}{k^2}\Gamma_{11} - \frac{k_x k_y}{k^2}\Gamma_{22} & \frac{k_y^2}{k^2}\Gamma_{11} + \frac{k_x^2}{k^2}\Gamma_{22} & \frac{k_y}{k}\Gamma_{13} \\ \frac{k_x}{k}\Gamma_{31} & \frac{k_y}{k}\Gamma_{31} & \Gamma_{33} \end{bmatrix}$$

Green's function regions



General result for the leading order term

Using solutions to the electric and the magnetic Green's function we can evaluate $I^{(2)}(k_x, \zeta, k_{1y}, k_{2y})$ as

$$
\begin{aligned}
- \; \frac{1}{k^2}\frac{1}{k'^2}\frac{1}{2\kappa}\frac{1}{2\kappa'} \Bigg[& \frac{1}{\Delta}\frac{1}{\Delta'} M(-\alpha_1, -\alpha_1') M(-\alpha_2, -\alpha_2') (k_x^2 + k_y k_y')^2 \zeta^4 \\
& + \frac{1}{\Delta}\frac{1}{\bar{\Delta}'} M(-\alpha_1, \bar{\alpha}_1') M(-\alpha_2, \bar{\alpha}_2') k_x^2 (k_y - k_y')^2 \zeta^2 \kappa'^2 \\
& + \frac{1}{\bar{\Delta}}\frac{1}{\Delta'} M(\bar{\alpha}_1, -\alpha_1') M(\bar{\alpha}_2, -\alpha_2') k_x^2 (k_y - k_y')^2 \zeta^2 \kappa^2 \\
+ \frac{1}{\bar{\Delta}}\frac{1}{\bar{\Delta}'} & \left\{ M(\bar{\alpha}_1, \bar{\alpha}_1')(k_x^2 + k_y k_y')\kappa\kappa' + M(-\bar{\alpha}_1, -\bar{\alpha}_1') k^2 k'^2 \frac{1}{\varepsilon_1} \right\} \\
& \times \left\{ M(\bar{\alpha}_2, \bar{\alpha}_2')(k_x^2 + k_y k_y')\kappa\kappa' + M(-\bar{\alpha}_2, -\bar{\alpha}_2') k^2 k'^2 \frac{1}{\varepsilon_2} \right\} \Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &= \Big[(1 - \alpha_1^2 e^{-2\kappa_1 d_1})(1 - \alpha_2^2 e^{-2\kappa_2 d_2})\, e^{\kappa a} \\
&\qquad - \alpha_1 \alpha_2 (1 - e^{-2\kappa_1 d_1})(1 - e^{-2\kappa_2 d_2})\, e^{-\kappa a} \Big], \\
M(\alpha_i, \alpha_i') &= (\varepsilon_i - 1) \Big[(1 - \alpha_i^2)\, e^{-\kappa_i d_i} (1 - \alpha_i'^2)\, e^{-\kappa_i' d_i} \\
&\qquad - (1 + \alpha_i)(1 - \alpha_i\, e^{-2\kappa_i d_i})(1 + \alpha_i')(1 - \alpha_i'\, e^{-2\kappa_i' d_i}) \Big],
\end{aligned}
$$

General result for the leading order term(cont.)

$\kappa_i^2 = k^2 + \zeta^2 \varepsilon_i$
$\bar{\kappa}_i = \kappa_i / \varepsilon_i$
$\alpha_i = (\kappa_i - \kappa)/(\kappa_i + \kappa)$
Quantities with primes are obtained by replacing $k_y \to k_y'$ everywhere.
Quatities with bar are defined in similar way with κ replaced with

$$\kappa_i \longrightarrow \overline{\kappa_i} = \frac{\kappa_i}{\epsilon_i}$$

Conductor limit

In the conductor limit $\epsilon \to \infty$

$$I^{(2)}_{\varepsilon\to\infty}(\kappa,\kappa',k_y-k'_y) = -\frac{\kappa}{\sinh\kappa a}\frac{\kappa'}{\sinh\kappa' a}\left[1+\frac{\{\kappa^2+\kappa'^2-(k_y-k'_y)^2\}^2}{4\,\kappa^2\kappa'^2}\right].$$

For the case of sinusoidal corrugations described by
$h_1(y) = h_1\sin[k_0(y+y_0)]$
$h_2(y) = h_2\sin[k_0 y]$
the lateral force is evaluated to be

$$F^{(2)}_{\varepsilon\to\infty} = 2k_0 a\,\sin(k_0 y_0)\left|F^{(0)}_{\text{Cas}}\right|\frac{h_1}{a}\frac{h_2}{a}A^{(1,1)}_{\varepsilon\to\infty}(k_0 a),$$

Conductor limit cont.

$$A^{(1,1)}_{\varepsilon\to\infty}(t_0) = \frac{15}{\pi^4}\int_{-\infty}^{\infty} dt \int_0^{\infty} \bar{s}\, d\bar{s}\, \frac{s}{\sinh s}\frac{s_+}{\sinh s_+}\left[\frac{1}{2} + \frac{(s^2+s_+^2-t_0^2)^2}{8\, s^2 s_+^2}\right],$$

where $s^2 = \bar{s}^2 + t^2$ and $s_+^2 = \bar{s}^2 + (t+t_0)^2$. The first term corresponds to the Dirichlet scalar case.

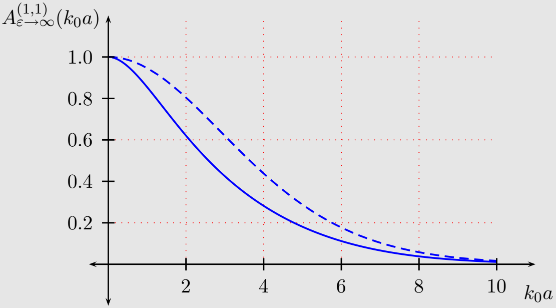


Figure: Plot of $A^{(1,1)}_{\varepsilon\to\infty}(k_0 a)$ versus $k_0 a$. The dotted curve represents 2 times the Dirichlet case.

Conductor limit cont.

We can evaluate one of the integrals to get

$$A^{(1,1)}_{\varepsilon\to\infty}(t_0) = \frac{15}{4}\int_0^\infty du\, \frac{\sin(2t_0 u/\pi)}{(2t_0 u/\pi)} \left[\frac{\sinh^2 u}{\cosh^6 u}\left(\frac{7}{2} - \sinh^2 u\right) - \frac{1}{2}\left(\frac{2t_0}{\pi}\right)^2 \frac{\sinh^2 u}{\cosh^4 u} + \frac{1}{16}\left(\frac{2t_0}{\pi}\right)^4 \frac{\sinh^2 u}{\cosh^2 u}\right], \tag{2}$$

which reproduces the result in Emig *et. al.* apart from an overall factor of 2.

The double integral representation is more useful for numerical evaluation because of the oscillatory nature of the function $\sin x/x$ the above expression.

Thin plate limit

It is under construction.......
It is interesting to analyze $d_i \to 0$ limit. If we set

$$\omega^2 (\epsilon_i - 1) = \omega_{p_i}^2$$
$$\lambda_i = \omega_{p_i}^2 d_i = \text{fixed as } d_i \to 0,$$

then we can write our potential as

$$V_i = \lambda_i \, \delta(z - a_i - h_i(y)).$$

This should model 2D sheets. However, if $a \sim 10^{-7} m$ and $\omega_p \sim 10^{15} Hz$, then for conductor case

$$\omega_{p_i}^2 da \gg 1 \Rightarrow d \gg 10^{-7} m,$$

which is not exactly a single atom thin layer. This limit can work for weak case.

Conclusion and things to do

1. We have obtained result for leading order contribution to the lateral Casimir force for corrugated dielectric slabs and obtained conductor limit for the same.
2. Obtain thin plate limit and get exact results in weak case.
3. Calculate next-to-leading order contribution and compare results with experiment as well as various numerical/analytical exact results (Mohideen, Reynaud, Emig, Marachevsky).

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2. Thank you all for listening.

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