## Calculation of Highly Oscillatory Integrals by Quadrature Methods

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## Outline



#### Motivation

- Study of Vacuum Energy
- Oscillatory Integrals
- Earlier Literature

#### 2 Our Results

- Main Results
- Implementation



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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

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## A model for Vacuum Energy

Our model of quantum vacuum energy density near the boundary has the form  $\lambda z^{\alpha}$ .



Figure: Steeply rising potential near the boundary

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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

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#### Why are we interested?

Spectral analysis of the rising potential gives Energy momentum tensor:

$$\overline{T}(z) = \frac{1}{\pi^3} \int_0^\infty d\rho \int_0^1 du \sqrt{1 - u^2} \cos(2z\rho u - 2\delta(\rho u))$$
(1)

where,

$$\delta(u) = \operatorname{ArcTan}\left(-u\left(\frac{\operatorname{AiryAi}(-u^2)}{\operatorname{AiryAi}'(-u^2)}\right)\right)$$

Study of Vacuum Energy Oscillatory Integrals Earlier Literature

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#### What does it look like?



Figure: The oscillatory cosine function

#### left:u goes from 100 to 100.0002 right: u goes from 100 to 100.004

Study of Vacuum Energy Oscillatory Integrals Earlier Literature

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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

# $\overline{T}(z)$ is highly oscillatory.

#### • Takes hours, if not days to calculate.

- Only for  $\alpha = 1$ .
- We need to check for higher values of  $\alpha$ .
- Similar  $\overline{T}(z)$  for higher  $\alpha$  values are bound to give more highly oscillatory integrals
  - We need systematic way to calculate these oscillatory integrals.
  - Check whether our model for potential is plausible.

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Our Results
 Main Results

Implementation



Study of Vacuum Energy Oscillatory Integrals Earlier Literature

## **Newton-Cotes Rule**

- Trapezoidal rule
- Simpson's rule



Figure: Plot showing integration by trapezoidal rule

$$\int_{a}^{b} f(x) dx \approx c_1 f(a) + c_2 f(b) = \frac{(b-a)}{2} (f(a) + f(b)).$$

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Our Results	Oscillatory Integrals
Summary	Earlier Literature

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$$\int_a^b f(x) dx \approx c_1 f(x_1) + c_2 f(x_2).$$

• Here,  $c_1, c_2, x_1$ , and  $x_2$  are all unknowns.

 In this case, these four constants are found by integrating third order polynomials and equating the coefficients.

$$x_1 = \frac{b-a}{2} \frac{-1}{\sqrt{3}} + \frac{b+a}{2},$$

$$x_2 = \frac{b-a}{2}\frac{1}{\sqrt{3}} + \frac{b+a}{2},$$

$$c_1 = \frac{b-a}{2}$$
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Our Results	Oscillatory Integrals
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Our Results	Oscillatory Integrals
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$$\int_{a}^{b} f(x) \sin \omega x dx \text{ and } \int_{0}^{\infty} \frac{f(x)}{x} \sin \omega x$$
$$\int f(x) \sin(\omega x) dx = \sum_{m=\mu}^{2\mu+2} f(x) \sin(\omega x)$$

 $m_{\mu}(x_{\mu}) = f(x_{\mu}), m_{\mu+1}(x_{\mu+1}) = f(x_{\mu+1}), \text{ and } m_{\mu+2}(x_{\mu+2}) = f(x_{\mu+2}).$ 

$$\int_{a}^{b} f(x) {
m sin} \omega x dx pprox \sum_{\mu=0}^{n-1} \int_{x_{2\mu}}^{x_{2\mu+2}} m_{\mu}(x) {
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Our Results	Oscillatory Integrals
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Our Results	Oscillatory Integrals
Summary	Earlier Literature

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Motivation	Study of Vacuum Energy
Our Results	Oscillatory Integrals
Summary	Earlier Literature

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Our Results	Oscillatory Integrals
Summary	Earlier Literature

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Motivation	Study of Vacuum Energy
Our Results	Oscillatory Integrals
Summary	Earlier Literature

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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

#### **ClenshawCurtis Method**

C.W.Clenshaw and A.R. Curtis in 1960. Expand f(x) in Chebyshev polynomials.

$$f(x) = F(t) = \frac{1}{2}a_0 + a_1T_1(t) + a_2T_2(t) + \dots + \frac{1}{2}a_nT_n(t), (a \le x \le b)$$
(2)

where,

$$T_n(t) = \cos(n \cos^{-1}(t), \ t = \frac{2x - (b + a)}{b - a}$$
(3)

and this eventually reduces to

$$f(x) = \frac{a_0}{2}T_0(x) + \sum_{n=1}^{\infty} a_n T_n(x), x_n = \cos(\frac{n\pi}{N}).$$
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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

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Motivation	Study of Vacuum Ene
Our Results	Oscillatory Integrals
Summary	Earlier Literature

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Study of Vacuum Energy Oscillatory Integrals Earlier Literature

#### Levin-Iserles' Method

#### Improvement over Filon's method

$$Q_{2}^{F}[f] = \left(-\frac{1}{i\omega} - 6\frac{1 + e^{i\omega}}{i\omega^{3}} + 12\frac{1 - e^{i\omega}}{\omega^{4}}\right)f(0)$$
(5)  
+  $\left(\frac{e^{i\omega}}{i\omega} + 6\frac{1 + e^{i\omega}}{i\omega^{3}} - 12\frac{1 - e^{i\omega}}{\omega^{4}}\right)f(1)$   
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Main Results Implementation

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Main Results Implementation

#### Was it worth the time?

#### Yes, and No.

- Iserles' method did not work for our  $\overline{T}(z)$  integral.
- Were able to calculate integrals much faster.
- Not very consistent.



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Implementation





## What quadrature method to choose?

- Levin-Iserles' method seems more promising.
- Clenshawcurtis' qudrature method also works well.



#### Figure: $\overline{T}(z)$ using Levin and Clenshawcurtis method

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- Highly oscillatory integrals can be calculated much faster than by conventional methods.
- choose methods judiciously.
- Reduce error for integrands with large frequency.

#### Outlook

- Not enough data for conclusion.
- Check for higher values of  $\alpha$ .

Krishna Thapa Department of Physics & Astronomy Texas A&M Calculation of Oscillatory Integrals by Quadrature Methods

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#### NSF-0554849 and PHY-0968269.



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## For Further Reading I

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