Distributions in Spaces with Thick Points II

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Preliminaries

We shall need to consider the differentiation of functions and distributions defined only on a smooth hypersurface $\Sigma$ of $\mathbb{R}^n$. Naturally, if $(v_\alpha)_{1 \leq \alpha \leq n-1}$ is a local Gaussian coordinate system and $f$ is defined on $\Sigma$ then one may consider the derivatives $\partial f / \partial v_\alpha$, $1 \leq \alpha \leq n - 1$. However, it is many times convenient and necessary to consider derivatives with respect to the variables $(x_j)_{1 \leq j \leq n}$ of the surrounding space $\mathbb{R}^n$. The $\delta-$derivatives are defined as follows.

**Definition**

Suppose $f$ is a smooth function defined in $\Sigma$ and let $F$ be any smooth extension of $f$ to an open neighborhood of $\Sigma$ in $\mathbb{R}^n$; the derivatives $\partial F / \partial x_j$ will exist, but their restriction to $\Sigma$ will depend not only on $f$ but also on the extension employed. However, it can be shown that the formulas

$$
\frac{\delta f}{\delta x_j} = \left( \frac{\partial F}{\partial x_j} - n_j \frac{dF}{dn} \right) \bigg|_\Sigma,
$$

(1)

where $n = (n_j)$ is the normal unit vector to $\Sigma$ and where $dF/dn = n_k \partial F / \partial x_k$ is the derivative of $F$ in the normal direction.
Preliminaries

Fact

Delta derivatives $\delta f / \delta x_j$, $1 \leq j \leq n$, that depend only on $f$ and not on the extension.
It can be shown that
\[ \frac{\delta f}{\delta x_j} = \frac{\partial f}{\partial v_\alpha} \frac{\partial v_\alpha}{\partial x_j}, \tag{2} \]

In this talk, we suppose now that the surface is \( S^{n-1} \), the unit sphere in \( \mathbb{R}^n \). Let \( f \) be a smooth function defined in \( S^{n-1} \), that is, \( f(\mathbf{w}) \) is defined if \( \mathbf{w} \in \mathbb{R}^n \) satisfies \( |\mathbf{w}| = 1 \). While (1) can be applied for any extension \( F \) of \( f \), the fact that our surface is \( S \) allows us to consider some rather natural extensions. In particular, there is an extension to \( \mathbb{R}^n \setminus \{0\} \) that is homogeneous of degree 0, namely,
\[ F_0(\mathbf{x}) = f \left( \frac{\mathbf{x}}{r} \right), \tag{3} \]

where \( r = |\mathbf{x}| \). Since \( dF_0/dn = 0 \) we obtain
\[ \frac{\delta f}{\delta x_j} = \left. \frac{\partial F_0}{\partial x_j} \right|_S. \tag{4} \]
\[
\frac{\delta (\phi \psi)}{\delta x_j} = \phi \frac{\delta \psi}{\delta x_j} + \frac{\delta \phi}{\delta x_j} \psi, \quad (5)
\]

**Lemma**

\[
\frac{\delta^T (\phi \psi)}{\delta x_j} = \phi \frac{\delta^T \psi}{\delta x_j} + \frac{\delta \phi}{\delta x_j} \psi, \quad (6)
\]

**Proof.**

\[
\left\langle \frac{\delta^T (\phi \psi)}{\delta x_j}, \zeta \right\rangle = - \left\langle \phi \psi, \frac{\delta \zeta}{\delta x_j} \right\rangle = - \left\langle \psi, \phi \frac{\delta \zeta}{\delta x_j} \right\rangle = - \left\langle \psi, \frac{\delta (\phi \zeta)}{\delta x_j} - \frac{\delta \phi}{\delta x_j} \zeta \right\rangle = \left\langle \phi \frac{\delta^T \psi}{\delta x_j} + \frac{\delta \phi}{\delta x_j} \psi, \zeta \right\rangle.
\]
Thus

\[
\frac{\delta^T \phi}{\delta x_j} = \frac{\delta^T (\phi \cdot 1)}{\delta x_j} = \phi \frac{\delta^T 1}{\delta x_j} + \frac{\delta \phi}{\delta x_j},
\]

And

\[
\frac{\delta^T 1}{\delta x_j} = -(n - 1) n_j
\]

So

\[
\frac{\delta^T n_i}{\delta x_j} = \delta_{ij} - (n_i n_j) - (n - 1)(n_i n_j) = \delta_{ij} - n - 1 (n_i n_j)
\]
A Short Review

Problem

\[ \int_{0}^{\infty} \cos(2kx) \, dx. \]

Proof.

[1]

\[ \int_{0}^{\infty} \cos(2kx) \, dx. = \frac{1}{2} \int_{-\infty}^{+\infty} \cos(2kx) \, dx \]

\[ = \frac{1}{2} \int_{-\infty}^{+\infty} e^{2ikx} \, dx \]

\[ = \pi \delta(2k) = \frac{\pi}{2} \delta(k) \]
Proof.

By definition of a distribution, we must evaluate this limit on a test function, $f(k)$, with support in $[0, \infty)$:

$$
\lim_{x \to \infty} \int_0^\infty \frac{\sin(2kx)}{2k} f(k) \, dk = \lim_{x \to \infty} \int_0^\infty \frac{\sin(2kx)}{2k} f(k) \, dk,
$$

$$
= \frac{1}{2} f(0) \int_0^\infty \frac{\sin(u)}{u} \, du = \frac{\pi}{4} f(0).
$$

So $\int_0^\infty \cos(2kx) \, dx = \frac{\pi}{4} \delta(k)$.
A Short Review

Solution

[Correction of proof 1]

\[
\int_{0}^{\infty} \cos(2kx) \, dx = \frac{1}{2} \int_{-\infty}^{+\infty} \cos(2kx) \, dx
\]
\[
= \frac{1}{2} \int_{-\infty}^{+\infty} e^{2ikx} \, dx = \frac{1}{2} \mathcal{F}\{\widetilde{1}; 2k\},
\]
\[
= \pi \tilde{\delta}(2k) = \frac{\pi}{2} \delta(k). \tag{7}
\]

Here

\[
\int_{-\infty}^{+\infty} e^{2ikx} \, dx = \mathcal{F}\{\widetilde{1}; 2k\} = 2\pi \tilde{\delta}(2k) = \pi \delta(k).
\]

is the Fourier transform of the funcion 1 in the space \( \mathcal{W}' \) and result in \( \mathcal{S}' \), the thick point space. This result holds for \( k \) positive or negative.
Solution

[Correction of proof 2]
As were pointed out, in solution 2, we have "secretly" multiplied $H(k)$.

$$\lim_{x \to \infty} \int_{-\infty}^{+\infty} \left( \frac{\sin(2kx)}{2k} H(k) \right) f(k) \, dk = \lim_{x \to \infty} \int_{0}^{\infty} \frac{\sin(2kx)}{2k} f(k) \, dk$$

In fact if we want the result for $k > 0$, we need to apply the projection multiplication operator $M'_H : S'_* \to S'$:

$$H(k) \int_{0}^{\infty} \cos(2kx) \, dx = \frac{\pi}{2} M'_H \left( \tilde{\delta}(k) \right) = \frac{\pi}{4} \delta(k) \quad (8)$$

Now the consistency of the results holds.
Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

**Definition**

A function $\phi$ defined on $\mathbb{R}^n$ is in $D_*,a(\mathbb{R}^n)$ iff

$$
\phi(a + x) = \phi(a + rw) \sim \sum_{J=N}^{\infty} a_J(w) r^J
$$

where $N$ is an integer, and $w \in S^{n-1}$, $a_J(w) \in D(S^{n-1})$.

Moreover, we require the asymptotic development to be "strong".

Namely, for any differentiation operator $(\partial/\partial x)^p = (\partial^{p_1} \ldots \partial^{p_n})/\partial x_1^{p_1} \ldots \partial x_n^{p_n}$, the asymptotic development of $(\partial/\partial x)^p \phi(x)$ exists and is equal to the term-by-term differentiation of $\sum_{J=N}^{\infty} a_J(w) r^J$. 
We use $\mathcal{D}_\ast (\mathbb{R}^n)$ to denote $\mathcal{D}_{\ast,0} (\mathbb{R}^n)$. 
Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

**Definition**

Define $D_*^{[k]}(\mathbb{R}^n)$ as the subspace consists of test functions

$$
\phi(rw) \sim \sum_{J=k}^{\infty} a_J(w) r^J
$$

Notice that $D_*^{[k]}(\mathbb{R}^n)$ is not closed under differentiation.

Note: In particular, if $\phi$ is a smooth function, then

$$
\phi(a + r\omega) \sim a_0 + \sum_{J=1}^{\infty} a_J(\omega) r^J \in D_*^{[0]}(\mathbb{R}^n). \text{ So } D_a(\mathbb{R}^n) \subset D_{*,a}(\mathbb{R}^n).
$$
The Topology—to have a TVS

**Definitions**

Define a seminorm.

\[ \|\phi\|_{l,m} = \sup_{|p| \leq m} \left( \frac{\left( \partial/ \partial x \right)^p \phi(x) - \sum_{j=N-|p|}^{l-1} a_{j,p}(w) r^j}{r^l} \right), \quad l \geq N - |p| \quad (9) \]

where

\[ (\partial/ \partial x)^p \phi(x) \sim \sum_{J=N-|p|}^{\infty} a_{J,p}(w) r^J \quad (10) \]

A sequence \( \{ \phi_\alpha \} \) in \( D_*(\mathbb{R}^n) \) converges to \( \varphi \) iff there exists an integer \( N \) such that \( \varphi \in D_*^{[N]}(\mathbb{R}^n) \), and a compact set \( K \) such that for any \( l, m \), we have \( \|\phi - \phi_\alpha\|_{l,m} \to 0 \) as \( \alpha \to \infty \).
Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

Fact

$\mathcal{D}(\mathbb{R}^n)$ is a subspace of $\mathcal{D}_*(\mathbb{R}^n)$

$$i : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}_*(\mathbb{R}^n)$$

$$\phi \mapsto \phi$$
Definitions

The space of distributions on $\mathbb{R}^n$ with a thick point is the dual space of that contains all the continuous linear functionals of the test functions. We denote it $\mathcal{D}'_*(\mathbb{R}^n)$.
Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Definitions

The space of distributions on $\mathbb{R}^n$ with a thick point is the dual space of that contains all the continuous linear functionals of the test functions. We denote it $\mathcal{D}'_*(\mathbb{R}^n)$.

Theorem

$$\mathcal{D} \xleftarrow{i} \mathcal{D}_*,a. \quad (12)$$

$$\mathcal{D}'_*,a \xrightarrow{\pi} \mathcal{D}'.$$  

$\pi$, the projection operator is given explicitly as

$$\langle \pi(f), \phi \rangle_{\mathcal{D}'\times\mathcal{D}} = \langle f, i(\phi) \rangle_{\mathcal{D}'_*,a\times\mathcal{D}_*,a}. \quad (13)$$
Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Theorem

The projection operator is given explicitly as

$$
\pi, \text{ the projection operator is given explicitly as }

\langle \pi (f), \phi \rangle_{\mathcal{D}' \times \mathcal{D}} = \langle f, i (\phi) \rangle_{\mathcal{D}'_* a \times \mathcal{D}'_* a}.
$$

Theorem

Let $g \in \mathcal{D}'$, there exists a distribution $f \in \mathcal{D}'_* a$, s.t. $\pi (f) = g$. 
Example

Suppose \( f(x) \) is a locally integrable function in \( \mathbb{R}^n \), homogeneous of degree 0. Now let's define a "thick delta function" \( f \delta_{\ast} \in \mathcal{D}'(\mathbb{R}^n) \):

Let \( \phi \) be a test function in \( \mathcal{D}(\mathbb{R}^n) \), thus by definition \( \phi \) could be asymptotically expanded as

\[
\sum_{J=N}^{\infty} a_J(w) r^J = a_N(w) r^N + \cdots + a_0(w) + a_1(w) r + \cdots
\]

Then \( f \delta_{\ast} \) is given by

\[
\langle f \delta_{\ast}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)} := \frac{1}{C_{n-1}} \langle f(w), a_0(w) \rangle_{\mathcal{D}'(S^{n-1}) \times \mathcal{D}(S^{n-1})} \quad (14)
\]

\[
= \frac{1}{C_{n-1}} \int_{S^{n-1}} f(w) a_0(w) d\sigma(w)
\]
Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Example

When $n = 3$,

$$
\langle f \delta_*, \phi \rangle = \frac{1}{4\pi} \int_{S^{n-1}} f(w) a_0(w) \, d\sigma(w)
$$

Example

In particular, if $f(x) \equiv 1$, then $f \delta_* = \delta_*$:

$$
\langle \delta_*, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)} = \frac{1}{C_{n-1}} \langle 1, a_0(w) \rangle_{\mathcal{D}'(S^{n-1}) \times \mathcal{D}(S^{n-1})}
$$

$$
= \frac{1}{C_{n-1}} \int_{S^{n-1}} a_0(w) \, d\sigma(w)
$$

We may call $\delta_*$ the "plain thick delta function".
Projection of a Thick Delta Function onto the Usual Distribution Space

Example

Since a usual test function $\psi \in \mathcal{D}'(\mathbb{R}^n)$ can be asymptotically expanded as its Taylor expansion: $\psi(rw) \sim a_0 + \sum_{J=1}^{\infty} a_J(w) r^J$, so

\[
\langle \pi(f \delta_*), \psi \rangle = \langle f \delta_*, i(\psi) \rangle
\]

\[
= \frac{1}{C_{n-1}} \int_{S^{n-1}} f(w) a_0 d\sigma(w)
\]

\[
= \frac{a_0}{C_{n-1}} \int_{S^{n-1}} f(w) d\sigma(w)
\]
Projection of a Thick Delta Function onto the Usual Distribution Space

Example

In particular, for a plain thick delta function, it projects onto the usual delta function:

\[ \left\langle \pi \left( \delta_* \right), \psi \right\rangle = \frac{a_0}{C_{n-1}} \int_{S^{n-1}} d\sigma(w) \]

\[ = a_0 = \phi(0) \]

\[ = \left\langle \delta, \psi \right\rangle \]
Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Definition

(thick delta functions of degree m) A thick delta functions of degree m, denoted $f \delta^{[m]}_\ast$, acting on a thick test function $\phi(x)$ is defined as the action of $f$ on $a_m(w)$ in the corresponding asymptotic expansion divide by the surface area of $S^{n-1}$. Namely,

$$\langle f \delta^{[m]}_\ast, \phi \rangle_{\mathcal{D}'_\ast(\mathbb{R}^n) \times \mathcal{D}_\ast(\mathbb{R}^n)} = \frac{1}{C_{n-1}} \langle f, a_m(w) \rangle_{\mathcal{D}'_\ast(S^{n-1}) \times \mathcal{D}_\ast(S^{n-1})}$$

Example

If $f$ is a locally integrable function in $\mathbb{R}^n$, homogeneous of degree 0, a natural example would be

$$\langle f \delta^{[m]}_\ast, \phi \rangle_{\mathcal{D}'_\ast(\mathbb{R}^n) \times \mathcal{D}_\ast(\mathbb{R}^n)} = \frac{1}{C_{n-1}} \langle f, a_m(w) \rangle_{\mathcal{D}'_\ast(S^{n-1}) \times \mathcal{D}_\ast(S^{n-1})} = \frac{1}{C_{n-1}} \int_{S^{n-1}} f(w) a_m(w) d\sigma(w)$$
Let $f, g \in \mathcal{D}'(\mathbb{R}^n)$, and $\phi(x) \in \mathcal{D}_*(\mathbb{R}^n)$ is a test function. We define the following algebraic operators:

1. $\langle f + g, \phi \rangle = \langle f, \phi \rangle + \langle g, \phi \rangle$.
2. $\langle f(Ax), \phi(x) \rangle = \frac{1}{|\det A|} \langle f(x), \phi(A^{-1}x) \rangle$, where $A$ is a non-singular $n \times n$ matrix. In particular, $\langle f(-x), \phi(x) \rangle = \langle f(x), \phi(-x) \rangle$.
3. $\langle f(x + c), \phi(x) \rangle = \langle f(x), \phi(x - c) \rangle$, where $c \in \mathbb{R}^n$.
4. $\langle \rho f, \phi \rangle = \langle f, \rho \phi \rangle$, where $\rho$ is a multiplier of $\mathcal{D}_{*,a}$, i.e. $\rho \phi \in \mathcal{D}_{*,a}$, $\forall \phi \in \mathcal{D}_{*,a}$.
Derivatives on Thick Distributions

Definition

The $p^{th}$ order derivative of a thick distribution $f \in \mathcal{D}'_*$ is given by

$$\left\langle \left( \frac{\partial^*}{\partial \mathbf{x}} \right)^p f, \phi \right\rangle = (-1)^{|p|} \left\langle f, \left( \frac{\partial}{\partial \mathbf{x}} \right)^p \phi \right\rangle = (-1)^{|p|} \left\langle f, \left( \frac{\partial^{p_1} \ldots \partial^{p_n}}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}} \right) \phi \right\rangle$$

We can call it "thick distributional derivative" to indicate the space $\mathcal{D}'_*$, in which $f$ sits.

Example

A first order partial derivative on $f$ may be given by

$$\left\langle \frac{\partial^* f}{\partial x_j}, \phi \right\rangle = - \left\langle f, \frac{\partial \phi}{\partial x_j} \right\rangle$$
Derivatives on Thick Distributions

Lemma

Suppose \( f \in \mathcal{D}' \); and the projection of \( f \) onto \( \mathcal{D}' \) is \( \pi(f) = g \). Then

\[
(\partial^*/\partial x)^p f = (\partial/\partial x)^p g.
\]

Proof.

if \( \phi \) is an ordinary test function that is in \( \mathcal{D}(\mathbb{R}^n) \); \( i \) denotes the inclusion map from \( \mathcal{D} \) to \( \mathcal{D}_* \); the projection of \( f \) from \( \mathcal{D}' \) to \( \mathcal{D}'_* \) is \( \pi(f) = g \), then we have,

\[
\left\langle \left( \frac{\partial^*}{\partial x} \right)^p f, i(\phi) \right\rangle = (-1)^{|p|} \left\langle f, \left( \frac{\partial}{\partial x} \right)^p \phi \right\rangle = (-1)^{|p|} \left\langle f, i \left[ \left( \frac{\partial}{\partial x} \right)^p \phi \right] \right\rangle
\]

\[
= (-1)^{|p|} \left\langle \pi(f), \left( \frac{\partial}{\partial x} \right)^p \phi \right\rangle = \left\langle \left( \frac{\partial}{\partial x} \right)^p g, \phi \right\rangle
\]
More about derivatives

Because $a_J(w)'s$ are finite on $\mathbb{S}^{n-1}$, the asymptotic expansion,

\[
\phi(rw) \sim \sum_{J=k}^{\infty} a_J(w) r^J = \sum_{J=k}^{\infty} a_J(x/r) r^J.\text{etc.}
\]

So

\[
\frac{\partial \phi}{\partial x_j} = \sum_{J=k}^{\infty} \frac{\partial a_J(x/r)}{\partial x_j} r^J + J a_J(x/r) n_j r^{-1}
\]
Lemma

The partial derivative of the thick delta function $w_i \delta^{[m]}_*$ with respect to $x_j$ is

$$\frac{\partial \left( n_i \delta_*^{[m]} \right)}{\partial x_j} = [\delta_{ij} + (-m - 1 - n) n_in_j] \delta_*^{[m+1]}$$

where $\delta_{ij}$ is the Kronecker delta function, $m$ is the degree of $w_i \delta_*^{[m]}$, $n$ is the dimension of $\mathbb{R}^n$, $w_i = x_i/r$, $w_i = x_i/r$.

Proof.

$$\left< \frac{\partial \left( n_i \delta_*^{[m]} \right)}{\partial x_j} , \phi \right> = - \left< n_i \delta_*^{[m]} , \frac{\partial \phi}{\partial x_j} \right>$$
Distributional Derivative of $1/r$

Proof.

\[
\begin{align*}
&= -\frac{1}{C_{n-1}} \int_{\mathbb{S}^{n-1}} n_i \left[ \frac{\delta a_{m+1}(w)}{\delta x_j} + (m + 1) n_j a_{m+1}(w) \right] d\sigma(w) \\
&= -\frac{1}{C_{n-1}} \left\langle n_i, \frac{\delta a_{m+1}(w)}{\delta x_j} \right\rangle_{\mathcal{D}'(\mathbb{S}^{n-1}) \times \mathcal{D}_*(\mathbb{S}^{n-1})} \\
&\quad - \frac{1}{C_{n-1}} \left\langle n_i, (m + 1) w_j a_{m+1}(w) \right\rangle_{\mathcal{D}'(\mathbb{S}^{n-1}) \times \mathcal{D}_*(\mathbb{S}^{n-1})} \\
&= \frac{1}{C_{n-1}} \left\langle \frac{\delta^T n_i}{\delta x_j}, a_{m+1}(w) \right\rangle_{\mathcal{D}'(\mathbb{S}^{n-1}) \times \mathcal{D}_*(\mathbb{S}^{n-1})} \\
&\quad - \left\langle (m + 1) n_i n_j \delta^{[m+1]}_*, \phi \right\rangle_{\mathcal{D}'(\mathbb{R}^n)}
\end{align*}
\]

Since

\[
\frac{\delta^T n_i}{\delta x_j} = \delta_{ij} - n(n_i n_j)
\]

The result is obtained.
In his paper, [4,Franklin] brought up a question: As a distribution, the well-known formula of the second derivative of $1/r$

$$\frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \left( \frac{4\pi}{3} \right) \delta_{ij} \delta(x)$$

cannot act on functions that are not smooth at the origin. In other words,

$$\frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \left( \frac{4\pi}{3} \right) \delta_{ij} \delta(x) \in \mathcal{D}' \text{ but } \notin \mathcal{D}'_*$$
Distributional Derivative of $1/r$

**Definition**

Let $\phi \in \mathcal{D}_* (\mathbb{R}^n)$,

$$\left\langle Pf \left( r^\lambda \right), \phi \right\rangle = F.p. \lim_{\varepsilon \to \infty} \int_{|x| \geq \varepsilon} r^\lambda \phi (x) \, dx$$

Hence

$$\left\langle \left( \frac{\partial^*}{\partial x} \right)^p Pf \left( r^\lambda \right), \phi \right\rangle = (-1)^{|p|} \left\langle Pf \left( r^\lambda \right), \left( \frac{\partial}{\partial x} \right)^p \phi \right\rangle$$

$$= (-1)^{|p|} F.p. \lim_{\varepsilon \to \infty} \int_{|x| \geq \varepsilon} r^\lambda \left( \frac{\partial}{\partial x} \right)^p \phi (x) \, dx$$
Lemma

We denote $\mathbb{S}_{\varepsilon}^{n-1}$ the $n$-1 sphere with radius $\varepsilon$.

Define $\langle r^\lambda n_j \delta (\mathbb{S}_{\varepsilon}^{n-1}), \phi (x) \rangle = \int_{\mathbb{S}_{\varepsilon}^{n-1}} \varepsilon^\lambda n_j \phi (x) \, dx$. For any thick test function $\phi$, then

$$\lim_{\varepsilon \to 0} \langle r^\lambda w_j \delta (\mathbb{S}_{\varepsilon}^{n-1}), \phi (x) \rangle = \begin{cases} 0 & \text{if } \lambda \notin \mathbb{Z} \\ = \langle C_{n-1} n_j \delta^{[1-n-\lambda]} [1], \phi (x) \rangle_{\mathcal{D}'(\mathbb{S}^{n-1}) \times \mathcal{D}(\mathbb{S}^{n-1})} & \text{if } \lambda \in \mathbb{Z} \end{cases}$$

Proof.

$$\langle r^\lambda n_j \delta (\mathbb{S}_{\varepsilon}^{n-1}), \phi (x) \rangle = \int_{\mathbb{S}_{\varepsilon}^{n-1}} \varepsilon^\lambda n_j \phi (x) \, dx$$

$$= \int_{\mathbb{S}^{n-1}} \varepsilon^\lambda n_j \phi (\varepsilon w) \varepsilon^{n-1} \, d\sigma (w)$$
Theorem

\[
\frac{\partial^*}{\partial x_j} \left( pf \left( r^\lambda \right) \right) = \begin{cases} 
\lambda x_j Pf \left( r^{\lambda-2} \right), & \lambda \notin \mathbb{Z} \\
\lambda x_j Pf \left( r^{\lambda-2} \right) + C_{n-1} n_j \delta_{*}^{[-\lambda-n+1]} & \lambda \in \mathbb{Z}
\end{cases}
\]  

(18)

where \( C_{n-1} \) is the surface area of the \( n - 1 \) unit sphere.
Distributional Derivative of \( r^\lambda \)

**Proof.**

By definition,

\[
\left\langle \frac{\partial^*}{\partial x_j} Pf \left( r^\lambda \right), \phi \right\rangle = - \left\langle Pf \left( r^\lambda \right), \frac{\partial \phi}{\partial x_j} \right\rangle = - \text{F.p. lim}_{\varepsilon \to \infty} \int_{|x| \geq \varepsilon} r^\lambda \frac{\partial \phi}{\partial x_j} \, dx
\]

(19)

\[
= \text{F.p. lim}_{\varepsilon \to \infty} \int_{|x| \geq \varepsilon} \frac{\overline{\partial} H (r - \varepsilon) r^\lambda}{\partial x_j} \phi \, dx = \left\langle \frac{\overline{\partial} H (r - \varepsilon) r^\lambda}{\partial x_j}, \phi \right\rangle
\]

We already know the usual distributional derivative of \( H (r - \varepsilon) r^\lambda \) is given by [2,Kanwal]

\[
\frac{\overline{\partial}}{\partial x_j} \left( H (r - \varepsilon) r^\lambda \right) = \lambda x_j r^{\lambda-2} H (r - \varepsilon) + r^\lambda n_j \delta (\mathcal{S}_\varepsilon)
\]
Distributional Derivative of $r^\lambda$

Proof.

So equation 19 becomes

$$F.p. \lim_{\varepsilon \to \infty} \left\langle \frac{\partial}{\partial x_j} \left( H(r - \varepsilon) r^\lambda \right), \phi \right\rangle$$

$$= F.p. \lim_{\varepsilon \to \infty} \left\langle \lambda x_j r^{\lambda-2} H(r - \varepsilon) + r^\lambda n_j \delta(S_\varepsilon), \phi \right\rangle$$

$$= \left\langle \lambda x_j Pf \left( r^{\lambda-2} \right), \phi \right\rangle + F.p. \lim_{\varepsilon \to \infty} \left\langle r^\lambda n_j \delta(S_\varepsilon), \phi \right\rangle$$

By 29, $\lim_{\varepsilon \to 0} \left\langle r^\lambda n_j \delta(S_\varepsilon^{n-1}), \phi(x) \right\rangle = C_{n-1} n_j \delta_{\ast}^{[1-n-\lambda]}$ So the theorem holds.

Example

When $n = 3, \lambda = -1$, the first derivative of $1/r$ is

$$\frac{\partial^*}{\partial x_j} \left( pf \left( r^{-1} \right) \right) = -x_j Pf \left( r^{-3} \right) + 4\pi n_j \delta_{\ast}^{[-1]}$$
Lemma

\[ x_j \delta^{[Q]}_* = w_j \delta^{[Q-1]}_*. \]

Theorem

If \( \lambda \) is an integer,

\[
\frac{\partial^2}{\partial x_j \partial x_k} \left( \text{pf} \left( r^\lambda \right) \right) = \delta_{jk} \lambda \text{pf} \left( r^{\lambda-2} \right) + \lambda (\lambda - 2) x_j x_k \text{pf} \left( r^{\lambda-4} \right)
+ C_{n-1} (2\lambda - 2) n_j n_k \delta^{[-\lambda-n+2]}_* + C_{n-1} \delta_{jk} \delta^{[-\lambda-n+2]}_*
\]
Second Order Distributional Derivative of $r^\lambda$

Proof.

Take the derivative of the first order derivative

$$\frac{\partial^*}{\partial x_j} \left( pf \left( r^\lambda \right) \right) = \lambda x_j Pf \left( r^{\lambda-2} \right) + C_{n-1} n_j \delta_*^{[-\lambda-n+1]}$$

we have

$$\frac{\partial^*}{\partial x_j \partial x_k} \left( pf \left( r^\lambda \right) \right) = \frac{\partial^* (x_j)}{\partial x_k} \lambda Pf \left( r^{\lambda-2} \right) + x_j \lambda \frac{\partial^* (Pf \left( r^{\lambda-2} \right))}{\partial x_k}$$

$$+ C_{n-1} \frac{\partial^* \left( n_j \delta_*^{[-\lambda-n+1]} \right)}{\partial x_k}$$

$$= \delta_{jk} \lambda Pf \left( r^{\lambda-2} \right) + \lambda x_j (\lambda - 2) Pf \left( r^{\lambda-4} \right)$$

$$+ \lambda x_j C_{n-1} n_k \delta_*^{[-\lambda-n+3]} + C_{n-1} \frac{\partial^* \left( n_j \delta_*^{[-\lambda-n+1]} \right)}{\partial x_k}$$

$\square$
Second Order Distributional Derivative of $1/r$

**Proof.**

Together with the lemma 27,
\[
\frac{\partial}{\partial x_k} \left( n_j \delta_*^{[-\lambda-n+1]} \right) = [\delta_{jk} + (\lambda - 2) n_j n_k] \delta_*^{[-\lambda-n+2]}.
\]

And by lemma 32, \( x_j \delta_*^{[-\lambda-n+3]} = n_j \delta_*^{[-\lambda-n+2]} \).

So,

**Theorem**

*If \( \lambda \) is an integer,*

\[
\frac{\partial^2}{\partial x_j \partial x_k} \left( \text{pf} \left( r^\lambda \right) \right) = \delta_{jk} \lambda \text{pf} \left( r^{\lambda-2} \right) + \lambda (\lambda - 2) x_j x_k \text{pf} \left( r^{\lambda-4} \right) + C_{n-1} (-n - 1) n_j n_k \delta_*^{[-\lambda-n+2]} + C_{n-1} \delta_{jk} \delta_*^{[-\lambda-n+2]}.
\]
Second Order Distributional Derivative of $1/r$

**Theorem**

If $\lambda$ is an integer,

\[
\frac{\partial^2}{\partial x_j \partial x_k} \left( pf \left( r^{\lambda} \right) \right) = \delta_{jk} \lambda Pf \left( r^{\lambda-2} \right) + \lambda (\lambda - 2) x_j x_k Pf \left( r^{\lambda-4} \right) \\
+ C_{n-1} (2\lambda - 2) n_j n_k \delta_{\ast}^{[-\lambda-n+2]} + C_{n-1} \delta_{jk} \delta_{\ast}^{[-\lambda-n+2]}
\]

**Example**

If $n = 3, \lambda = -1$,

\[
\frac{\partial^2}{\partial x_j \partial x_k} \left( pf \left( r^{-1} \right) \right) = 3x_j x_k Pf \left( r^{-5} \right) - \delta_{jk} Pf \left( r^{-3} \right) \\
- 16\pi n_j n_k \delta_{\ast} + 4\pi \delta_{jk} \delta_{\ast}
\]
Second Order Distributional Derivative of $1/r$

So in the thick point spaces,

$$
\frac{\partial^*}{\partial x_j \partial x_k} \left( \frac{1}{r} \right) = \frac{3x_j x_k - r^2 \delta_{jk}}{r^5} - \left( \frac{16\pi x_j x_k \delta^* (x)}{r^2} \right) + 4\pi \delta_{jk} \delta^* \quad (20)
$$

In particular, if $\phi (x) \in \mathcal{D} (\mathbb{R}^n)$, i.e., $a_0$ is a constant, then

$$
- \int_{\mathbb{R}^3} \frac{16\pi x_j x_k}{r^2} \delta^* (x) \phi (x) \, dx = -4a_0 \int_{\mathbb{S}^2} n_i n_j \, d\sigma (\omega) + 4\delta_{jk} a_0 = -a_0 \frac{16\pi}{3} \delta_{jk}
$$

$$
\int_{\mathbb{R}^3} 4\pi \delta_{jk} \delta^* (x) \phi (x) \, dx = 4\pi \delta_{jk} a_0
$$

Hence the projection is given explicitly as:

$$
\pi \left( \frac{\partial^*}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) \right) = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \left( \frac{4\pi}{3} \right) \delta_{ij} \delta (x) \quad (21)
$$
Second Order Distributional Derivative of $1/r$

Conclusion:

\[
\frac{\partial^*}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \left( \frac{16\pi x_j x_k \delta_*(x)}{r^2} \right) \frac{3}{4\pi} \delta_{ij} \delta(x) + 4\pi \delta_{jk} \delta_*
\]

is a thick distribution.

\[
\pi \left( \frac{\partial^*}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) \right) = \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} - \left( \frac{4\pi}{3} \right) \delta_{ij} \delta(x)
\]

is a usual distribution.
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Thank you!