Distributions in Spaces with Thick Points II

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Preliminaries

We shall need to consider the differentiation of functions and distributions defined only on a smooth hypersurface Σ of R^n. Naturally, if $(v_\alpha)_{1\le\alpha\le n-1}$ is a local Gaussian coordinate system and f is defined on Σ then one may consider the derivatives $\partial f/\partial v_\alpha$, $1 \le \alpha \le n-1$. However, it is many times convenient and necessary to consider derivatives with respect to the variables $(x_j)_{1\le j\le n}$ of the surrounding space R^n. The $\delta-$derivatives are defined as follows.

Definition

Suppose f is a smooth function defined in Σ and let F be any smooth extension of f to an open neighborhood of Σ in R^n; the derivatives $\partial F/\partial x_j$ will exist, but their restriction to Σ will depend not only on f but also on the extension employed. However, it can be shown that the formulas

$$\frac{\delta f}{\delta x_j} = \left(\frac{\partial F}{\partial x_j} - n_j \frac{dF}{dn}\right)\bigg|_{\Sigma}, \tag{1}$$

where $n = (n_j)$ is the normal unit vector to Σ and where $dF/dn = n_k \partial F/\partial x_k$ is the derivative of F in the normal direction.

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Fact

Delta derivatives $\delta f/\delta x_j$, $1 \leq j \leq n$, *that depend only on* f *and not on the extension.*

It can be shown that
$$\frac{\delta f}{\delta x_j}=\frac{\partial f}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial x_j}\,, \tag{2}$$

In this talk, we suppose now that the surface is $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$. Let f be a smooth function defined in $\mathbb{S}^{n-1}$, that is, $f(\mathbf{w})$ is defined if $\mathbf{w}\in\mathbb{R}^n$ satisfies $|\mathbf{w}|=1$. While (1) can be applied for any extension F of f, the fact that our surface is $\mathbb{S}$ allows us to consider some rather natural extensions. In particular, there is an extension to $\mathbb{R}^n\setminus\{\mathbf{0}\}$ that is homogeneous of degree 0, namely,

$$F_0(\mathbf{x})=f\left(\frac{\mathbf{x}}{r}\right)\,, \tag{3}$$

where $r=|\mathbf{x}|$. Since $dF_0/dn=0$ we obtain

$$\frac{\delta f}{\delta x_j}=\left.\frac{\partial F_0}{\partial x_j}\right|_{\mathbb{S}}\,. \tag{4}$$

Preliminaries

$$\frac{\delta(\phi\psi)}{\delta x_j} = \phi\frac{\delta\psi}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\psi\,, \tag{5}$$

Lemma

$$\frac{\delta^T(\phi\psi)}{\delta x_j} = \phi\frac{\delta^T\psi}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\psi\,, \tag{6}$$

Proof.

$$\begin{aligned}\left\langle \frac{\delta^T(\phi\psi)}{\delta x_j}, \zeta \right\rangle &= -\left\langle \phi\psi, \frac{\delta\zeta}{\delta x_j} \right\rangle = -\left\langle \psi, \phi\frac{\delta\zeta}{\delta x_j} \right\rangle \\ &= -\left\langle \psi, \frac{\delta(\phi\zeta)}{\delta x_j} - \frac{\delta\phi}{\delta x_j}\zeta \right\rangle = \left\langle \phi\frac{\delta^T\psi}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\psi, \zeta \right\rangle\,.\end{aligned}$$

□

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Thus

$$\frac{\delta^T \phi}{\delta x_j} = \frac{\delta^T (\phi \cdot 1)}{\delta x_j} = \phi \frac{\delta^T 1}{\delta x_j} + \frac{\delta \phi}{\delta x_j},$$

- And

$$\frac{\delta^T 1}{\delta x_j} = -(n-1)\, n_j$$

So

$$\frac{\delta^{\mathsf{T}} n_i}{\delta x_j} = \delta_{ij} - (n_i n_j) - (n-1)(n_i n_j) = \delta_{ij} - n - 1\, (n_i n_j)$$

A Short Review

Problem

$$\int_0^\infty \cos(2kx)\,dx.$$

Proof.

[1]

$$\begin{aligned}\int_0^\infty \cos(2kx)\,dx. &= \frac{1}{2}\int_{-\infty}^{+\infty} \cos(2kx)\,dx \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} e^{2ikx}\,dx \\ &= \pi\delta(2k) = \frac{\pi}{2}\delta(k)\end{aligned}$$

□

A Short Review

On the other hand,

Proof.

[2]

$$\int_0^\infty \cos(2kx)\,dx = \left.\frac{\sin(2kx)}{2k}\right|_{x=0}^{\infty} = \lim_{x\to\infty} \frac{\sin(2kx)}{2k}.$$

By definition of a distribution, we must evaluate this limit on a test function, $f(k)$, with support in $[0, \infty)$:

$$\lim_{x\to\infty} \int_0^\infty \frac{\sin(2kx)}{2k} f(k)\,dk = \lim_{x\to\infty} \int_0^\infty \frac{\sin(2kx)}{2k} f(k)\,dk,$$

$$= \frac{1}{2} f(0) \int_0^\infty \frac{\sin(u)}{u}\,du = \frac{\pi}{4} f(0).$$

So $\int_0^\infty \cos(2kx)\,dx = \frac{\pi}{4}\delta(k)$. □

A Short Review

Solution

[Correction of proof 1]

$$\int_0^{\infty} \cos(2kx)\, dx. = \frac{1}{2}\int_{-\infty}^{+\infty} \cos(2kx)\, dx$$
$$= \frac{1}{2}\int_{-\infty}^{+\infty} e^{2ikx}\, dx = \frac{1}{2}\mathcal{F}\{\widetilde{1}; 2k\},$$
$$= \pi\widetilde{\delta}(2k) = \frac{\pi}{2}\widetilde{\delta}(k). \tag{7}$$

Here

$$\int_{-\infty}^{+\infty} e^{2ikx}\, dx = \mathcal{F}\{\widetilde{1}; 2k\} = 2\pi\widetilde{\delta}(2k) = \pi\widetilde{\delta}(k).$$

is the Fourier transform of the funcion 1 *in the space* $\mathcal{W}'$ *and result in* $\mathcal{S}'_*$, *the thick point space. This result holds for* k *positive or negative.*

A Short Review

Solution

[Correction of proof 2]
As were pointed out, in solution 2, we have "secretly" multiplied $H(k)$.

$$\lim_{x\to\infty}\int_{-\infty}^{+\infty}\left(\frac{\sin(2kx)}{2k}H(k)\right)f(k)\,dk=\lim_{x\to\infty}\int_{0}^{\infty}\frac{\sin(2kx)}{2k}f(k)\,dk$$

In fact if we want the result for $k>0$, we need to apply the projection multiplicaiton operator $M'_H:\mathcal{S}'_*\to\mathcal{S}'$:

$$H(k)\int_{0}^{\infty}\cos(2kx)\,dx=\frac{\pi}{2}M'_H\left(\widetilde{\delta}(k)\right)=\frac{\pi}{4}\delta(k) \tag{8}$$

Now the consistency of the results holds.

Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

Definition

A function ϕ defined on $\mathbb{R}^n$ is in $\mathcal{D}_{*,a}(\mathbb{R}^n)$ iff

$$\phi(a+\mathbf{x}) = \phi(a+r\mathbf{w}) \sim \sum_{J=N}^{\infty} a_J(\mathbf{w})\, r^J$$

where N is an integer, and $\mathbf{w} \in \mathbb{S}^{n-1}$, $a_J(\mathbf{w}) \in \mathcal{D}\left(\mathbb{S}^{n-1}\right)$.
Moreover, we require the the asymptotic development to be "strong".
Namely, for any differentiation operator $(\partial/\partial\mathbf{x})^{\mathbf{p}} = (\partial^{p_1}...\partial^{p_n})/\partial x_1^{p_1}...\partial x_n^{p_n}$, the asymptotic development of $(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi(\mathbf{x})$ exists and is equal to the term-by-term differentiation of $\sum_{J=N}^{\infty} a_J(\mathbf{w})\, r^J$.

We use $\mathcal{D}_{*}\left(\mathbb{R}^{n}\right)$ to denote $\mathcal{D}_{*,0}\left(\mathbb{R}^{n}\right)$.

Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

Definition

Define $D_*^{[k]}(\mathbb{R}^n)$ as the subspace consists of test functions

$$\phi(r\mathbf{w}) \sim \sum_{J=k}^{\infty} a_J(\mathbf{w}) r^J$$

Notice that $D_*^{[k]}(\mathbb{R}^n)$ is not closed under differentiation.

Note: In particular, if ϕ is a smooth function, then
$\phi(a + r\boldsymbol{\omega}) \sim a_0 + \sum_{J=1}^{\infty} a_J(\boldsymbol{\omega}) r^J \in D_*^{[0]}(\mathbb{R}^n)$. So $\mathcal{D}_{\mathbf{a}}(\mathbb{R}^n) \subset \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$.

The Topology–to have a TVS

Definitions

Define a seminorm.

$$||\phi||_{l,m} = \sup_{|\mathbf{p}|\leq m} \frac{\left| (\partial/\partial\mathbf{x})^{\mathbf{p}} \phi(\mathbf{x}) - \sum_{j=N-|\mathbf{p}|}^{l-1} a_{j,\mathbf{p}}(\mathbf{w}) r^{j} \right|}{r^{l}}, l \geq N - |\mathbf{p}| \tag{9}$$

where

$$(\partial/\partial\mathbf{x})^{\mathbf{p}} \phi(\mathbf{x}) \sim \sum_{J=N-|\mathbf{p}|}^{\infty} a_{J,\mathbf{p}}(\mathbf{w}) r^{J} \tag{10}$$

A sequence $\{\phi_\alpha\}$ in $\mathcal{D}_*(\mathbb{R}^n)$ converges to φ iff there exists an integer N such that $\varphi \in D_*^{[N]}(\mathbb{R}^n)$,and a compact set K such that for any l, m, we have $||\phi - \phi_\alpha||_{l,m} \to 0$ as $\alpha \to \infty$.

Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

Fact

$\mathcal{D}(\mathbb{R}^n)$ is a subspace of $\mathcal{D}_*(\mathbb{R}^n)$

$$\begin{aligned} i \quad : \quad & \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}_*(\mathbb{R}^n) \\ & \phi \hookrightarrow \phi \end{aligned} \tag{11}$$

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Definitions

The space of distributions on $\mathbb{R}^n$ with a thick point is the dual space of that contains all the continous linear functionals of the test functions. We denote it $\mathcal{D}'_*(\mathbb{R}^n)$.

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Definitions

The space of distributions on $\mathbb{R}^n$ with a thick point is the dual space of that contains all the continous linear functionals of the test functions. We denote it $\mathcal{D}'_{*}\left(\mathbb{R}^n\right)$.

Theorem

$$\mathcal{D} \overset{i}{\hookrightarrow} \mathcal{D}_{*,a}. \tag{12}$$

$$\mathcal{D}'_{*,a} \overset{\pi}{\rightarrow} \mathcal{D}'.$$

π, the projection operator is given explicitly as

$$\langle \pi\left(f\right), \phi \rangle_{\mathcal{D}' \times \mathcal{D}} = \langle f, i\left(\phi\right) \rangle_{\mathcal{D}'_{*,a} \times \mathcal{D}_{*,a}}. \tag{13}$$

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

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Theorem

Let $g \in \mathcal{D}'$, there exists a distribution $f \in \mathcal{D}'_{*,a}$, s.t. $\pi(f) = g$.

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Example

Suppose $f(\mathbf{x})$ is a locally integrable funcion in $\mathbb{R}^n$, homogeneous of degree 0 Now let's define a "thick delta function" $f\delta_* \in \mathcal{D}'_*(\mathbb{R}^n)$:

Let ϕ be a test function in $\mathcal{D}_*(\mathbb{R}^n)$, thus by definition ϕ could be asymptotically expanded as

$\sum_{J=N}^{\infty} a_J(\mathbf{w}) r^J = a_N(\mathbf{w}) r^N + \ldots + a_0(\mathbf{w}) + a_1(\mathbf{w}) r + \ldots$

Then $f\delta_*$ is given by

$$\begin{aligned} \langle f\delta_*, \phi\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} &:= \frac{1}{C_{n-1}} \langle f(\mathbf{w}), a_0(\mathbf{w})\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1})\times\mathcal{D}_*(\mathbb{S}^{n-1})} \qquad (14) \\ &= \frac{1}{C_{n-1}} \int_{\mathbb{S}^{n-1}} f(\mathbf{w})\, a_0(\mathbf{w})\, d\sigma(\mathbf{w}) \end{aligned}$$

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Example

When $n = 3$,

$$\langle f\delta_*, \phi\rangle = \frac{1}{4\pi}\int_{\mathbb{S}^{n-1}} f(\mathbf{w})\, a_0(\mathbf{w})\, d\sigma(\mathbf{w})$$

Example

In particular, if $f(\mathbf{x}) \equiv 1$, then$f\delta_* = \delta_*$:

$$\begin{aligned}\langle \delta_*, \phi\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} \quad : &= \frac{1}{C_{n-1}}\langle 1, a_0(\mathbf{w})\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1})\times\mathcal{D}_*(\mathbb{S}^{n-1})} \\ &= \frac{1}{C_{n-1}}\int_{\mathbb{S}^{n-1}} a_0(\mathbf{w})\, d\sigma(\mathbf{w})\end{aligned}$$

. We may call δ_* the "plain thick delta function".

Projection of a Thick Delta Function onto the Usual Distribution Space

Example

Since a usual test function $\psi \in \mathcal{D}'(\mathbb{R}^n)$ can be asymptotically expanded as it's Taylor expansion: $\psi(r\mathbf{w}) \sim \mathbf{a}_0 + \sum_{J=1}^{\infty} \mathbf{a}_J(\mathbf{w}) r^J$, so

$$
\begin{aligned}
\langle \pi(f\delta_*), \psi \rangle &= \langle f\delta_*, i(\psi) \rangle && (15) \\
&= \frac{1}{C_{n-1}} \int_{\mathbb{S}^{n-1}} f(\mathbf{w}) a_0 d\sigma(\mathbf{w}) \\
&= \frac{a_0}{C_{n-1}} \int_{\mathbb{S}^{n-1}} f(\mathbf{w}) d\sigma(\mathbf{w}) && (16)
\end{aligned}
$$

Projection of a Thick Delta Function onto the Usual Distribution Space

Example

In particular, for a plain thick delta function, it projects onto the usual delta function:

$$\begin{aligned} \langle \pi(\delta_*), \psi \rangle &= \frac{a_0}{C_{n-1}} \int_{\mathbb{S}^{n-1}} d\sigma(\mathbf{w}) \\ &= a_0 = \phi(0) \\ &= \langle \delta, \psi \rangle \end{aligned} \tag{17}$$

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Definition

(thick delta functions of degree m) A thick delta functions of degree m, denoted $f\delta_*^{[m]}$, acting on a thick test function $\phi(\mathbf{x})$ is defined as the action of f on $a_m(\mathbf{w})$ in the corresponding asymptotic expansion divide by the surface area of $\mathbb{S}^{n-1}$. Namely,

$$\left\langle f\delta_*^{[m]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C_{n-1}} \left\langle f, a_m(\mathbf{w}) \right\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1})\times\mathcal{D}_*(\mathbb{S}^{n-1})}$$

Example

If f is a locally integrable funcion in $\mathbb{R}^n$, homogeneous of degree 0, a natural example would be

$$\begin{aligned}\left\langle f\delta_*^{[m]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} &= \frac{1}{C_{n-1}} \left\langle f, a_m(\mathbf{w}) \right\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1})\times\mathcal{D}_*(\mathbb{S}^{n-1})} \\ &= \frac{1}{C_{n-1}} \int_{\mathbb{S}^{n-1}} f(\mathbf{w})\, a_m(\mathbf{w})\, d\sigma(\mathbf{w})\end{aligned}$$

Space of Distributions on $\mathbb{R}^n$ with a Thick Point

Definitions

Let $f, g \in \mathcal{D}'_*(\mathbb{R}^n)$,and $\phi(\mathbf{x}) \in \mathcal{D}_*(\mathbb{R}^n)$ is a test function. We define the following algebraic operators:

1. $\langle f + g, \phi \rangle = \langle f, \phi \rangle + \langle g, \phi \rangle$.
2. $\langle f(A\mathbf{x}), \phi(\mathbf{x}) \rangle = \frac{1}{|\det A|} \left\langle f(\mathbf{x}), \phi\left(A^{-1}\mathbf{x}\right) \right\rangle$. where A is a non-singular $n \times n$ matrix. In particular, $\langle f(-\mathbf{x}), \phi(\mathbf{x}) \rangle = \langle f(\mathbf{x}), \phi(-\mathbf{x}) \rangle$
3. $\langle f(\mathbf{x} + \mathbf{c}), \phi(\mathbf{x}) \rangle = \langle f(\mathbf{x}), \phi(\mathbf{x} - \mathbf{c}) \rangle$, where $\mathbf{c} \in \mathbb{R}^n$.
4. $\langle \rho f, \phi \rangle = \langle f, \rho\phi \rangle$, where ρ is a multiplier of $\mathcal{D}_{*,a}$, i.e. $\rho\phi \in \mathcal{D}_{*,a}$, $\forall \phi \in \mathcal{D}_{*,a}$

Derivatives on Thick Distributions

Definition

The $\mathbf{p}-th$ order derivative of a thick distribution $f \in \mathcal{D}'_*$ is given by

$$\left\langle \left(\frac{\partial^*}{\partial \mathbf{x}}\right)^{\mathbf{p}} f, \phi \right\rangle = (-1)^{|\mathbf{p}|} \left\langle f, \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi \right\rangle = (-1)^{|\mathbf{p}|} \left\langle f, \frac{\left(\partial^{p_1} ... \partial^{p_n}\right) \phi}{\partial x_1^{p_1} ... \partial x_n^{p_n}} \right\rangle \quad ,\phi$$

We can call it "thick distributional derivative" to indicate the space $\mathcal{D}'_*$, in which f sits.

Example

a first order parital derivative on f may be given by

$$\left\langle \frac{\partial^* f}{\partial x_j}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x_j} \right\rangle$$

Derivatives on Thick Distributions

Lemma

Suppose $f \in \mathcal{D}'_*$; *and the projection of* f *onto* $\mathcal{D}'$ *is* $\pi(f) = g$. *Then* $(\partial^*/\partial\mathbf{x})^{\mathbf{p}} f = (\overline{\partial}/\partial\mathbf{x})^{\mathbf{p}} g$.

Proof.

if ϕ is an ordinary test function that is in $\mathcal{D}(\mathbb{R}^n)$; i denotes the inclusion map from $\mathcal{D}$ to $\mathcal{D}_*$; the projection of f from $\mathcal{D}'$to $\mathcal{D}'_*$ is $\pi(f) = g$, then we have,

$$\begin{aligned} \left\langle \left(\frac{\partial^*}{\partial \mathbf{x}}\right)^{\mathbf{p}} f, i(\phi) \right\rangle &= (-1)^{|\mathbf{p}|} \left\langle f, \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi \right\rangle = (-1)^{|\mathbf{p}|} \left\langle f, i\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi\right] \right\rangle \\ &= (-1)^{|\mathbf{p}|} \left\langle \pi(f), \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi \right\rangle = \left\langle \left(\frac{\overline{\partial}}{\partial \mathbf{x}}\right)^{\mathbf{p}} g, \phi \right\rangle \end{aligned}$$

□

More about derivatives

Because $a_J(\mathbf{w})' s$ are finite on $\mathbb{S}^{n-1}$, the asymptotic expansion,
$\phi(r\mathbf{w}) \sim \sum_{J=k}^{\infty} a_J(\mathbf{w}) r^J = \sum_{J=k}^{\infty} a_J(\mathbf{x}/r) r^J$.etc.
So

$$\frac{\partial \phi}{\partial x_j} = \sum_{J=k}^{\infty} \frac{\partial a_J(\mathbf{x}/r)}{\partial x_j} r^J + J a_J(\mathbf{x}/r) n_j r^{j-1}$$

Distributional Derivative of $1/r$

Lemma

The partial derivative of the thick delta function $w_i\delta_*^{[m]}$ with respect to x_j is

$$\frac{\partial\left(n_i\delta_*^{[m]}\right)}{\partial x_j} = \left[\delta_{ij} + (-m-1-n)\,n_i n_j\right]\delta_*^{[m+1]}$$

where δ_{ij} is the Kronecker delta function, m is the degree of $w_i\delta_*^{[m]}$, n is the dimension of $\mathbb{R}^n$, $w_i = x_i/r, w_i = x_i/r$.

Proof.

$$\left\langle \frac{\partial\left(n_i\delta_*^{[m]}\right)}{\partial x_j}, \phi \right\rangle = -\left\langle n_i\delta_*^{[m]}, \frac{\partial\phi}{\partial x_j} \right\rangle$$

□

Distributional Derivative of $1/r$

Proof.

$$= -\frac{1}{C_{n-1}} \int_{\mathbb{S}^{n-1}} n_i \left[\frac{\delta a_{m+1}(\mathbf{w})}{\delta x_j} + (m+1) n_j a_{m+1}(\mathbf{w}) \right] d\sigma(\mathbf{w})$$

$$= -\frac{1}{C_{n-1}} \left\langle n_i, \frac{\delta a_{m+1}(\mathbf{w})}{\delta x_j} \right\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1}) \times \mathcal{D}_*(\mathbb{S}^{n-1})}$$

$$- \frac{1}{C_{n-1}} \langle n_i, (m+1) w_j a_{m+1}(\mathbf{w}) \rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1}) \times \mathcal{D}_*(\mathbb{S}^{n-1})}$$

$$= \frac{1}{C_{n-1}} \left\langle \frac{\delta^{\mathsf{T}} n_i}{\delta x_j}, a_{m+1}(\mathbf{w}) \right\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1}) \times \mathcal{D}_*(\mathbb{S}^{n-1})} - \left\langle (m+1) n_i n_j \delta_*^{[m+1]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^{n-}}$$

Since

$$\frac{\delta^{\mathsf{T}} n_i}{\delta x_j} = \delta_{ij} - n(n_i n_j)$$

The result is obtained. □

Distributional Derivative of r^{λ}

In his paper, [4,Franklin] brought up a question: As a distribution, the well-known formula of the second derivative of $1/r$

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta\left(\mathbf{x}\right)$$

cannot act on functions that are not smooth at the orign.
In other words,

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta\left(\mathbf{x}\right) \in \mathcal{D}' but \notin \mathcal{D}'_*$$

Distributional Derivative of $1/r$

Definition

Let $\phi \in \mathcal{D}_* (\mathbb{R}^n)$,

$$\left\langle Pf\left(r^\lambda\right), \phi\right\rangle = F.p. \lim_{\varepsilon \to \infty} \int_{|\mathbf{x}| \geq \varepsilon} r^\lambda \phi(\mathbf{x})\, d\mathbf{x}$$

Hence

$$\begin{aligned} \left\langle \left(\frac{\partial^*}{\partial \mathbf{x}}\right)^{\mathbf{p}} Pf\left(r^\lambda\right), \phi\right\rangle &= (-1)^{|\mathbf{p}|} \left\langle Pf\left(r^\lambda\right), \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi\right\rangle \\ &= (-1)^{|\mathbf{p}|} F.p. \lim_{\varepsilon \to \infty} \int_{|\mathbf{x}| \geq \varepsilon} r^\lambda \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi(\mathbf{x})\, d\mathbf{x} \end{aligned}$$

Distributional Derivative of r^λ

Lemma

We denote $\mathbb{S}_\varepsilon^{n-1}$ *the n-1 sphere with radius* ε.
Define $\left\langle r^\lambda n_j \delta\left(\mathbb{S}_\varepsilon^{n-1}\right), \phi(\mathbf{x})\right\rangle = \int_{\mathbb{S}_\varepsilon^{n-1}} \varepsilon^\lambda n_j \phi(\mathbf{x})\, d\mathbf{x}$.*for any thick test function* ϕ,*then*

$$\lim_{\varepsilon\to 0}\left\langle r^\lambda w_j \delta\left(\mathbb{S}_\varepsilon^{n-1}\right), \phi(\mathbf{x})\right\rangle$$

$$\begin{cases} = 0 & \text{if } \lambda \notin \mathbb{Z} \\ = \left\langle C_{n-1} n_j \delta_*^{[1-n-\lambda]}, \phi(\mathbf{x})\right\rangle_{\mathcal{D}'_*(\mathbb{S}^{n-1})\times\mathcal{D}_*(\mathbb{S}^{n-1})} & \text{if } \lambda \in \mathbb{Z} \end{cases}$$

Proof.

$$\begin{aligned} \left\langle r^\lambda n_j \delta\left(\mathbb{S}_\varepsilon^{n-1}\right), \phi(\mathbf{x})\right\rangle &= \int_{\mathbb{S}_\varepsilon^{n-1}} \varepsilon^\lambda n_j \phi(\mathbf{x})\, d\mathbf{x} \\ &= \int_{S^{n-1}} \varepsilon^\lambda n_j \phi(\varepsilon\mathbf{w})\, \varepsilon^{n-1} d\sigma(\mathbf{w}) \end{aligned}$$

Distributional Derivative of r^λ

Theorem

$$\frac{\partial^*}{\partial x_j}\left(pf\left(r^\lambda\right)\right) = \begin{cases} \lambda x_j Pf\left(r^{\lambda-2}\right), & \lambda \notin \mathbb{Z} \\ \lambda x_j Pf\left(r^{\lambda-2}\right) + C_{n-1} n_j \delta_*^{[-\lambda-n+1]} & \lambda \in \mathbb{Z} \end{cases} \quad (18)$$

where C_{n-1} is the surface area of the $n-1$ unit sphere.

Distributional Derivative of r^λ

Proof.

By definition,

$$\left\langle \frac{\partial^*}{\partial x_j} Pf\left(r^\lambda\right), \phi \right\rangle = -\left\langle Pf\left(r^\lambda\right), \frac{\partial \phi}{\partial x_j} \right\rangle = -F.p. \lim_{\varepsilon \to \infty} \int_{|\mathbf{x}| \geq \varepsilon} r^\lambda \frac{\partial \phi}{\partial x_j} d\mathbf{x} \tag{19}$$

$$= F.p. \lim_{\varepsilon \to \infty} \int_{|\mathbf{x}| \geq \varepsilon} \frac{\overline{\partial} H\left(r-\varepsilon\right) r^\lambda}{\partial x_j} \phi d\mathbf{x} = \left\langle \frac{\overline{\partial} H\left(r-\varepsilon\right) r^\lambda}{\partial x_j}, \phi \right\rangle$$

We already know the usual distributional derivative of $H\left(r-\varepsilon\right) r^\lambda$ is given by [2,Kanwal]

$$\frac{\overline{\partial}}{\partial x_j}\left(H\left(r-\varepsilon\right) r^\lambda\right) = \lambda x_j r^{\lambda-2} H\left(r-\varepsilon\right) + r^\lambda n_j \delta\left(\mathbb{S}_\varepsilon\right)$$

□

Distributional Derivative of r^λ

Proof.

So equation 19 becomes

$$F.p.\lim_{\varepsilon\to\infty}\left\langle \frac{\overline{\partial}}{\partial x_j}\left(H\left(r-\varepsilon\right)r^\lambda\right),\phi\right\rangle$$
$$= F.p.\lim_{\varepsilon\to\infty}\left\langle \lambda x_j r^{\lambda-2}H\left(r-\varepsilon\right)+r^\lambda n_j\delta\left(\mathbb{S}_\varepsilon\right),\phi\right\rangle$$
$$= \left\langle \lambda x_j Pf\left(r^{\lambda-2}\right),\phi\right\rangle + F.p.\lim_{\varepsilon\to\infty}\left\langle r^\lambda n_j\delta\left(\mathbb{S}_\varepsilon\right),\phi\right\rangle$$

By 29, $\lim_{\varepsilon\to0}\left\langle r^\lambda n_j\delta\left(\mathbb{S}_\varepsilon^{n-1}\right),\phi\left(\mathbf{x}\right)\right\rangle = C_{n-1}n_j\delta_*^{[1-n-\lambda]}$ So the theorem holds. □

Example

When $n=3,\lambda=-1$,the first derivative of $1/r$ is

$$\frac{\partial^*}{\partial x_j}\left(pf\left(r^{-1}\right)\right) = -x_j Pf\left(r^{-3}\right)+4\pi n_j\delta_*^{[-1]}$$

Second Order Distributional Derivative of r^λ

Lemma

$x_j \delta_*^{[Q]} = w_j \delta_*^{[Q-1]}.$

Theorem

If λ is an integer,

$$\frac{\partial^{*2}}{\partial x_j \partial x_k}\left(pf\left(r^\lambda\right)\right) = \delta_{jk}\lambda Pf\left(r^{\lambda-2}\right) + \lambda\left(\lambda-2\right)x_j x_k Pf\left(r^{\lambda-4}\right) + C_{n-1}\left(2\lambda-2\right)n_j n_k \delta_*^{[-\lambda-n+2]} + C_{n-1}\delta_{jk}\delta_*^{[-\lambda-n+2]}$$

Second Order Distributional Derivative of r^λ

Proof.

Take the derivative of the first order derivative $\frac{\partial^*}{\partial x_j}\left(pf\left(r^\lambda\right)\right) = \lambda x_j Pf\left(r^{\lambda-2}\right) + C_{n-1} n_j \delta_*^{[-\lambda-n+1]}$, we have

$$\begin{aligned}
\frac{\partial^{*2}}{\partial x_j \partial x_k}\left(pf\left(r^\lambda\right)\right) &= \frac{\partial^*\left(x_j\right)}{\partial x_k}\lambda Pf\left(r^{\lambda-2}\right) + x_j\lambda\frac{\partial^*\left(Pf\left(r^{\lambda-2}\right)\right)}{\partial x_k} \\
&\quad + C_{n-1}\frac{\partial^*\left(n_j\delta_*^{[-\lambda-n+1]}\right)}{\partial x_k} \\
&= \delta_{jk}\lambda Pf\left(r^{\lambda-2}\right) + \lambda x_j\left(\lambda-2\right)Pf\left(r^{\lambda-4}\right) \\
&\quad + \lambda x_j C_{n-1} n_k \delta_*^{[-\lambda-n+3]} + C_{n-1}\frac{\partial^*\left(n_j\delta_*^{[-\lambda-n+1]}\right)}{\partial x_k}
\end{aligned}$$

□

Second Order Distributional Derivative of $1/r$

Proof.

Together with the lemma 27,
$\frac{\partial\left(n_j\delta_*^{[-\lambda-n+1]}\right)}{\partial x_k} = \left[\delta_{jk} + (\lambda-2)\, n_j n_k\right] \delta_*^{[-\lambda-n+2]}$.
And by lemma 32, $x_j\delta_*^{[-\lambda-n+3]} = n_j\delta_*^{[-\lambda-n+2]}$.
So,

□

Theorem

If λ is an integer,

$$\frac{\partial^{*2}}{\partial x_j\partial x_k}\left(pf\left(r^{\lambda}\right)\right) = \delta_{jk}\lambda Pf\left(r^{\lambda-2}\right) + \lambda(\lambda-2)\, x_j x_k\, Pf\left(r^{\lambda-4}\right)$$
$$+C_{n-1}(-n-1)\, n_j n_k \delta_*^{[-\lambda-n+2]} + C_{n-1}\delta_{jk}\delta_*^{[-\lambda-n+2]}$$

Second Order Distributional Derivative of $1/r$

Theorem

If λ is an integer,

$$\begin{aligned}\frac{\partial^{*2}}{\partial x_j \partial x_k}\left(pf\left(r^{\lambda}\right)\right) &= \delta_{jk}\lambda Pf\left(r^{\lambda-2}\right)+\lambda\left(\lambda-2\right)x_j x_k Pf\left(r^{\lambda-4}\right)\\ &\quad +C_{n-1}\left(2\lambda-2\right)n_j n_k \delta_*^{[-\lambda-n+2]}+C_{n-1}\delta_{jk}\delta_*^{[-\lambda-n+2]}\end{aligned}$$

Example

If $n=3, \lambda=-1$,

$$\begin{aligned}\frac{\partial^{*2}}{\partial x_j \partial x_k}\left(pf\left(r^{-1}\right)\right) &= 3x_j x_k Pf\left(r^{-5}\right)-\delta_{jk}Pf\left(r^{-3}\right)\\ &\quad -16\pi n_j n_k \delta_* + 4\pi\delta_{jk}\delta_*\end{aligned}$$

Second Order Distributional Derivative of $1/r$

So in the thick point spaces,

$$\frac{\partial^*}{\partial x_j \partial x_k}\left(\frac{1}{r}\right) = \frac{3x_j x_k - r^2\delta_{jk}}{r^5} - \left(\frac{16\pi x_j x_k \delta_*(\mathbf{x})}{r^2}\right) + 4\pi\delta_{jk}\delta_* \tag{20}$$

In particular, if $\phi(\mathbf{x}) \in \mathcal{D}(\mathbb{R}^n)$, *i.e.*, a_0 is a constant, then

$$-\int_{\mathbb{R}^3} \frac{16\pi x_j x_k}{r^2}\delta_*(\mathbf{x})\,\phi(\mathbf{x})\,\mathrm{d}x = -4a_0\int_{\mathbb{S}^2} n_i n_j \mathrm{d}\sigma(\boldsymbol{\omega}) + 4\delta_{jk}a_0 = -a_0\frac{16\pi}{3}\delta_{jk}$$

$$\int_{\mathbb{R}^3} 4\pi\delta_{jk}\delta_*(\mathbf{x})\,\phi(\mathbf{x})\,\mathrm{d}x = 4\pi\delta_{jk}a_0$$

Hence the projection is given explicitly as:

$$\pi\left(\frac{\partial^*}{\partial x_i \partial x_j}\left(\frac{1}{r}\right)\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta(\mathbf{x}) \tag{21}$$

Second Order Distributional Derivative of $1/r$

Conclusion:

$$\frac{\partial^*}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{16\pi x_j x_k \delta_*(\mathbf{x})}{r^2}\right) + 4\pi\delta_{jk}\delta_* \tag{22}$$

is a thick distribution.

$$\pi\left(\frac{\partial^*}{\partial x_i \partial x_j}\left(\frac{1}{r}\right)\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta(\mathbf{x}) \tag{23}$$

is a usual distribution.

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Thank you!