**Exercise**

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad 0 < x < \infty
\]

\[
u(x, y, 0) = f(x, y) \quad 0 < y < L \quad t > 0
\]

\[
\frac{\partial u}{\partial x} (0, y, t) = g(y)
\]

\[
u(x, 0, t) = u(x), \quad u(x, L, t) = 0
\]

\[
U = V + W \quad V = V_1 + V_2
\]

\[
V_i - BC
\]

\[
\frac{\partial u}{\partial x} (0, y) = 0, \quad V_1(x, 0) = \lambda(x), \quad V_1(x, L) = 0
\]

\[
V_2 - BC
\]

\[
\frac{\partial u}{\partial x} (0, y) = g(y), \quad V_2(x, 0) = 0, \quad V_2(x, L) = 0
\]

\[
W = U - V
\]

\[
W - BC
\]

\[
W(x, y, 0) = f(x, y) - V_1(x, y, 0) - V_2(x, y, 0)
\]

\[
\frac{\partial W}{\partial x} (0, y, t) = 0
\]

\[
W(x, 0, t) = 0
\]

\[
W(x, L, t) = 0
\]

**How To Solve Each:**

**V₁:** The nonhomogeneous boundary condition, \(V₁(x, 0) = \lambda(x)\) goes to infinity, so it must be solved using a Fourier transform. The boundary condition \(\frac{\partial u}{\partial x} (0, y) = 0\) will cause this to be a cosine transform.
$V_2$: The nonhomogeneous boundary condition 
\[ \frac{\partial V_2}{\partial x}(0,y) = g(y) \] is on a finite interval 
\((0 < y < L)\), so it will be solved by 
using a Fourier series.

The boundary condition of \(V_2(x,0) = 0\) 
will cause this to be a sine series.

$W$: With \(V = V_1 + V_2\) solved for the nonhomogeneous 
boundary conditions we can then 
apply this to \(W = U - V\) which will 
have all homogeneous boundary conditions. 
Separation of variables can be used to 
find the solution which will be 
a combination of a Fourier transform 
and series.
\[ \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \quad 0 < x < \infty \]
\[ w(x, y, 0) = k(x, y) \quad 0 < y < L \]
\[ w(x, 0, t) = w(x, H, t) = 0 \quad 0 < t < \infty \]
\[ \frac{\partial w}{\partial x} (0, y, t) = 0 \quad (\text{o.d}) \]

\[ F(\omega_1, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty k(x, y) \cos(\omega_1 x) \, dx \quad \text{used in (1)} \]

\[ k(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega_1, y) \cos(\omega_1 x) \, d\omega_1 \quad \text{used in (1)} \]

\[ \Phi(\omega_1, \omega_2) = \frac{2}{H} \int_0^H \int_0^\infty F(\omega_1, y) \sin \left( \frac{\omega_2 \pi x}{H} \right) \, dy \, dx \quad \text{used in (2)} \]

\[ F(\omega_1, y) = \sum_{\omega_2} \Phi(\omega_1, \omega_2) \sin \left( \frac{\omega_2 \pi y}{H} \right) \quad \text{used in (2)} \]

Combining above formulas:

\[ \Phi(\omega_1, \omega_2) = \frac{2}{H} \sqrt{\frac{2}{\pi}} \int_0^H \int_0^\infty k(x, y) \cos(\omega_1 x) \sin \left( \frac{\omega_2 \pi x}{H} \right) \, dx \, dy \]

\[ k(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sum_{\omega_2} \Phi(\omega_1, \omega_2) \sin \left( \frac{\omega_2 \pi x}{H} \right) \cos(\omega_1 x) \, d\omega_1 \]
\[ \tilde{\omega}^2 = \omega_1^2 + \omega_2^2 \]
\[ \frac{\partial \tilde{W}}{\partial t} = -\tilde{\omega}^2 \tilde{W} \quad (3) \]

\[ \Phi(\omega_1, \omega_2) = \tilde{W}(\omega_1, \omega_2, t) = \]
\[ \frac{2}{H} \sqrt{\frac{2}{\pi}} \int_0^H \int_0^{\infty} W(x, y, t) \cos(\omega_1 x) \sin\left(\frac{\omega_2 \pi y}{H}\right) \, dx \, dy \]

Solution of (3): \[ \tilde{W}(\omega_1, \omega_2, t) = A(\omega_1, \omega_2) e^{-\sigma^2 t} \]

Therefore final solution:
\[ A(\omega_1, \omega_2) = \tilde{W}(\omega_1, \omega_2, 0) = \]
\[ \frac{2}{H} \sqrt{\frac{2}{\pi}} \int_0^H \int_0^{\infty} k(x, y) \cos(\omega_1 x) \sin\left(\frac{\omega_2 \pi y}{H}\right) \, dx \, dy \]

\[ W(x, y, t) = \]
\[ \sqrt{\frac{2}{\pi}} \int_0^H \int_0^{\infty} A(\omega_1, \omega_2) e^{-\sigma^2 t} \sin\left(\frac{\omega_2 \pi y}{H}\right) \cos(\omega_1 x) \, d\omega_1 \]