EXERCISE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad 0 < x < \infty$$

$$u(x,y,0) = f(x,y) \qquad 0 < y < L \qquad 0 < t < \infty$$

$$\frac{\partial u}{\partial x}(0,y,t) = g(y)$$

$$u(x,0,t) = h(x), \quad u(x,L,t) = 0$$

$U = V + W \qquad V = V_1 + V_2$

V_1 - BC

$$\frac{\partial V_1}{\partial x}(0,y) = 0, \quad V_1(x,0) = h(x), \quad V_1(x,L) = 0$$

V_2 - BC

$$\frac{\partial V_2}{\partial x}(0,y) = g(y), \quad V_2(x,0) = 0, \quad V_2(x,L) = 0$$

$W = U - V$

W - BC

$$W(x,y,0) = f(x,y) - V_1(x,y,0) - V_2(x,y,0)$$

$$\frac{\partial W}{\partial x}(0,y,t) = 0$$

$$W(x,0,t) = 0$$

$$W(x,L,t) = 0$$

HOW TO SOLVE EACH:

V_1: The nonhomogeneous boundary condition, $V_1(x,0) = h(x)$ goes to infinity, so it must be solved using a Fourier transform.

The boundary condition $\frac{\partial V_1}{\partial x}(0,y) = 0$ will cause this to be a cosine transform

V_2: The nonhomogeneous boundary condition $\frac{\partial V_2}{\partial x}(0,y) = g(y)$ is on a finite interval $(0<y<L)$, so it will be solved by using a Fourier series.
The boundary condition of $V_2(x,0)=0$ will cause this to be a sine series

W: With $V = V_1 + V_2$ solved for the nonhomogeneous boundary conditions we can then apply this to $W = U - V$, which will have all homogeneous boundary conditions.
Seperation of variables can be used to find the solution which will be a combination of a Fourier transform and series.

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

$0 < x < \infty$
$0 < y < L$
$0 < t < \infty$

$$w(x,y,0) = k(x,y)$$

$$w(x,0,t) = w(x,H,t) = 0 \quad (2)$$

$$\frac{\partial w}{\partial x}(0,y,t) = 0 \quad (1)$$

$$F(\omega_1, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty k(x,y)\cos(\omega_1 \cdot x)\,\delta x$$

$$k(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega_1, y)\cos(\omega_1 \cdot x)\,\delta \omega_1$$

used cos b/c of (1)

$$\mathcal{F}(\omega_1, \omega_2) = \frac{2}{H} \int_0^H F(\omega_1, y)\sin\left(\frac{\omega_2 \pi y}{H}\right)\delta y$$

$$F(\omega_1, y) = \sum_{\omega_2 = 1}^{\infty} \mathcal{F}(\omega_1, \omega_2)\sin\left(\frac{\omega_2 \pi y}{H}\right)$$

used sin b/c of (2)

combining above formulas:

$$\mathcal{F}(\omega_1, \omega_2) = \frac{2}{H}\sqrt{\frac{2}{\pi}} \int_0^H \int_0^\infty k(x,y)\cos(\omega_1 \cdot x)\sin\left(\frac{\omega_2 \cdot \pi \cdot y}{H}\right)\delta x\,\delta y$$

$$k(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sum_{\omega_2 = 1}^{\infty} \mathcal{F}(\omega_1, \omega_2)\sin\left(\frac{\omega_2 \cdot \pi \cdot y}{H}\right)\cos(\omega_1 \cdot x)\,\delta \omega_1$$

$$\tilde{\omega}^2 = \omega_1^2 + \omega_2^2$$

$$\frac{\partial \bar{W}}{\partial t} = -\tilde{\omega}^2 \bar{W} \quad (3)$$

$$\bar{\mathcal{F}}(\omega_1, \omega_2) = \bar{W}(\omega_1, \omega_2, t) =$$

$$\frac{2}{H}\sqrt{\frac{2}{\pi}} \int_0^H \int_0^\infty W(x,y,t)\cos(\omega_1 \cdot x)\sin\left(\frac{\omega_2 \pi y}{H}\right)\delta x\,\delta y$$

solution of (3): $\bar{W}(\omega_1, \omega_2, t) = A(\omega_1, \omega_2)e^{-\tilde{\omega}^2 t}$

Therefore final solution:

$$A(\omega_1, \omega_2) = \bar{W}(\omega_1, \omega_2, 0) =$$

$$\frac{2}{H}\sqrt{\frac{2}{\pi}} \int_0^H \int_0^\infty K(x,y)\cos(\omega_1 x)\sin\left(\frac{\omega_2 \pi y}{H}\right)\delta x\,\delta y$$

$$W(x,y,t) =$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \sum_{\omega_2=1}^{\infty} A(\omega_1, \omega_2)e^{-\tilde{\omega}^2 t}\sin\left(\frac{\omega_2 \pi y}{H}\right)\cos(\omega_1 \cdot x)\,\delta\omega_1$$

5