Math 148 Exam III Practice Problems

This review should not be used as your sole source of preparation for the exam. You should also re-work all examples from the lecture notes and all suggested homework problems.

1. Find the largest possible domain and corresponding range of each function. Describe the level curves \( f(x, y) = k \) of each function and determine the possible values of \( k \).

(a) \( f(x, y) = \frac{2}{x + y} \)
(b) \( f(x, y) = \sqrt{36 - 9x^2 - 4y^2} \)

2. Draw a contour map showing several level curves of each function.

(a) \( f(x, y) = xy \)
(b) \( f(x, y) = x^2 + 9y^2 \)

3. Evaluate the limit

\[
\lim_{{(x,y) \to (0,0)}} \frac{3x^2 - y^2}{x^2 + y^2},
\]

if it exists.

4. Evaluate the limit

\[
\lim_{{(x,y) \to (0,0)}} \frac{2xy}{x^3 + xy},
\]

if it exists.

5. Consider the function \( f \) defined by

\[
f(x, y) = \begin{cases} 3xy, & \text{if } (x, y) \neq (0, 0) \\ \frac{x^2 + y^2}{x^2+y^2}, & \text{if } (x, y) = (0, 0) \\ 0, & \end{cases}
\]

Determine whether \( f \) is continuous at \((0, 0)\). Explain your answer.

6. Find all first-order partial derivatives of each function.

(a) \( f(x, y) = \frac{x^2}{y} - \frac{y^2}{x} \)
(b) \( f(x, y) = e^{xy^2} \)
(c) \( f(x, y) = y \ln(xy) \)

7. Find all first-order partial derivatives of each function.

(a) \( f(x, y, z) = x^3y^2z + \frac{x}{yz} \)
(b) \( f(x, y, z) = e^{y \cos z} \sin x \)
8. Find all second-order partial derivatives of each function.
   (a) \( f(x, y) = x^2 y + x \sqrt{y} \)
   (b) \( f(x, y) = \sin(x + y) + \cos(x - y) \)

9. Find an equation of the tangent plane to \( f(x, y) = \sqrt{4 - x^2 - 2y^2} \) at \((1, -1)\).

10. Find the linearization of \( f(x, y) = y \cos(x - y) \) at \((2, 2)\) and use it to approximate \( f(1.9, 2.1) \). Check the accuracy of your approximation using a calculator.

11. Find the linearization of \( \mathbf{F}(x, y) = (e^{x^2-y^2}, \ln(2x - y^2)) \) at \((1, 1)\).

12. Let \( f(x, y) = e^x \sin y \), where \( x(t) = t^2 \) and \( y(t) = 4t \). Find \( \frac{df}{dt} \) when \( t = 0 \).

13. Compute the directional derivative of \( f(x, y) = \sqrt{xy - 2x^2} \) at the point \( P = (1, 6) \) in the direction of the point \( Q = (3, 1) \).

14. In what direction does \( f(x, y) = \sqrt{x^2 - y^2} \) increase most rapidly at \((5, 3)\)? What is the largest increase?

15. Find a unit vector that is normal to the level curve of the function \( f(x, y) = x^2 + \frac{y^2}{4} \) at \((1, 2)\).

16. Find all local extrema and saddle points of \( f(x, y) = -x^2 - y^2 + 6x + 8y - 21 \).

17. Find all local extrema and saddle points of \( f(x, y) = 2x^4 + y^2 - 12xy \).

18. Find the absolute extrema of \( f(x, y) = x^2 - y^2 + 4x + y \) on the rectangular region \( R = \{(x, y)| -4 \leq x \leq 0, 0 \leq y \leq 1\} \)

19. Find the absolute extrema of \( f(x, y) = x^2 + 3y^2 + 2y \) on the disk \( D = \{(x, y)| x^2 + y^2 \leq 1\} \)

20. Consider the linear system
\[
\begin{align*}
  x_1(t + 1) &= -0.4x_1(t) + 0.2x_2(t) \\
  x_2(t + 1) &= -0.3x_1(t) + 0.1x_2(t)
\end{align*}
\]
Determine the stability of \( \mathbf{x} = 0 \).

21. Consider the linear system
\[
\begin{align*}
  x_1(t + 1) &= -0.2x_1(t) - 0.4x_2(t) \\
  x_2(t + 1) &= 0.6x_1(t) + 0.1x_2(t)
\end{align*}
\]
Determine the stability of \( \mathbf{x} = 0 \).
22. Find all nonnegative equilibria of
\[ x_1(t + 1) = 2x_1(t)[1 - x_1(t)] \]
\[ x_2(t + 1) = x_1(t)[1 - x_2(t)] \]
and determine their stability.

23. Find all biologically-relevant equilibria of the Nicholson-Bailey model
\[ N_{t+1} = 4N_t e^{-0.1P_t} \]
\[ P_{t+1} = N_t [1 - e^{-0.1P_t}] \]
and analyze their stability.
1. Find the largest possible domain and corresponding range of each function. Describe the level curves \( f(x, y) = k \) of each function and determine the possible values of \( k \).

(a) \( f(x, y) = \frac{2}{x + y} \)

The expression for \( f \) is defined provided that the denominator is not zero. So the domain of \( f \) is

\[ D = \{ (x, y) \mid x + y \neq 0 \} \]

Note that \( x + y \neq 0 \) means that points on the line \( y = -x \) must be excluded from the domain.

For every point \((x, y)\) in the domain \(D\), either (i) \(0 < x + y < \infty\) in which case

\[ 0 < \frac{2}{x + y} < \infty \]

or (ii) \(-\infty < x + y < 0\) in which case

\[ -\infty < \frac{2}{x + y} < 0 \]

So the range of \( f \) is \((-\infty, 0) \cup (0, \infty)\).

The level curves are defined by \( f(x, y) = k \), where \( k \) is a constant in the range of \( f \). Therefore,

\[
\begin{align*}
\frac{2}{x + y} &= k \\
2 &= k(x + y) \\
\frac{2}{k} &= x + y \\
\frac{2}{k} - x &= y
\end{align*}
\]

The level curves are lines where \( k \neq 0 \).
(b) \( f(x, y) = \sqrt{36 - 9x^2 - 4y^2} \)

The expression for \( f \) is defined provided that the quantity under the square root sign is nonnegative. So the domain of \( f \) is

\[
D = \{ (x, y) \mid 9x^2 + 4y^2 \leq 36 \} = \{ (x, y) \mid \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \}
\]

For every point \((x, y)\) in the domain \(D\),

\[
0 \leq 36 - 9x^2 - 4y^2 < 36
\]

Taking the positive square root, we have

\[
0 \leq \sqrt{36 - 9x^2 - 4y^2} < 6
\]

So the range of \( f \) is \([0, 6]\).

The level curves are defined by \( f(x, y) = k \), where \( k \) is a constant in the range of \( f \). Therefore,

\[
\sqrt{36 - 9x^2 - 4y^2} = k
\]

\[
36 - 9x^2 - 4y^2 = k^2
\]

\[
36 - k^2 = 9x^2 + 4y^2
\]

The level curves are ellipses where \( 0 \leq k \leq 6 \).
2. Draw a contour map showing several level curves of each function.

(a) \( f(x, y) = xy \)

The level curves are defined by

\[ xy = k, \]

where \( k \) is a constant in the range of \( f \).

If \( k = 0 \), then \( x = 0 \) or \( y = 0 \). Thus, the level curve for \( k = 0 \) consists of the coordinate axes \( x = 0 \) and \( y = 0 \). If \( k \neq 0 \), then the level curves are defined by

\[ y = \frac{k}{x}. \]

This is a family of hyperbolas centered at \((0, 0)\). A contour plot showing the level curves for \( k = 0, \pm1, \pm4, \pm8 \) is shown below.

(b) \( f(x, y) = x^2 + 9y^2 \)

The level curves are defined by

\[ x^2 + 9y^2 = k, \]

where \( k \) is a constant in the range of \( f \).

This is a family of concentric ellipses with center \((0, 0)\). A contour plot showing the level curves for \( k = 1, 9, 18 \) is shown below.
3. Evaluate the limit
\[ \lim_{(x,y) \to (0,0)} \frac{3x^2 - y^2}{x^2 + y^2}, \]
if it exists.

Along the x-axis \((y = 0)\),
\[ \lim_{(x,y) \to (0,0)} \frac{3x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{3x^2}{x^2} = \lim_{x \to 0} 3 = 3. \]

Along the y-axis \((x = 0)\),
\[ \lim_{(x,y) \to (0,0)} \frac{3x^2 - y^2}{x^2 + y^2} = \lim_{y \to 0} \frac{-y^2}{y^2} = \lim_{y \to 0} -1 = -1 \neq 3. \]

Therefore, the limit does not exist.

4. Evaluate the limit
\[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^3 + xy}, \]
if it exists.

Along the x-axis \((y = 0)\),
\[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^3 + xy} = \lim_{x \to 0} \frac{0}{x^3} = \lim_{x \to 0} 0 = 0. \]

Along the parabola \(y = x^2\),
\[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^3 + xy} = \lim_{x \to 0} \frac{2x^3}{2x^3} = \lim_{x \to 0} 1 = 1 \neq 0. \]

Therefore, the limit does not exist.
5. Consider the function $f$ defined by

$$f(x, y) = \begin{cases} 
\frac{3xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\
0, & \text{if } (x, y) = (0, 0)
\end{cases}$$

Determine whether $f$ is continuous at $(0,0)$. Explain your answer.

The function $f$ is defined at $(0,0)$ as $f(0,0) = 0$. However, we will show that the limit of the function as $(x,y) \to (0,0)$ does not exist.

Along the $x$-axis ($y=0$),

$$\lim_{(x,y) \to (0,0)} \frac{3xy}{x^2 + y^2} = \lim_{x \to 0} \frac{0}{x^2} = \lim_{x \to 0} 0 = 0.$$ 

Along the line $y = x$,

$$\lim_{(x,y) \to (0,0)} \frac{3xy}{x^2 + y^2} = \lim_{x \to 0} \frac{3x^2}{2x^2} = \lim_{x \to 0} \frac{3}{2} = \frac{3}{2} \neq 0.$$ 

Therefore, the limit does not exist. Thus, $f$ is discontinuous at $(0,0)$.

6. Find the first-order partial derivatives of each function.

(a) $f(x, y) = \frac{x^2}{y} - \frac{y^2}{x}$

For simplicity, we rewrite the function as

$$f(x, y) = \frac{x^3 - y^3}{xy}$$

Using the Quotient Rule, we have

$$f_x(x, y) = \frac{3x^2y - y(x^3 - y^3)}{x^2y^2} = \frac{2x^3y + y^4}{x^2y^2} = \frac{2x^3 + y^3}{x^2y}$$

$$f_y(x, y) = \frac{-3y^2(x) - x(x^3 - y^3)}{x^2y^2} = \frac{-2xy^3 - x^4}{x^2y^2} = \frac{-2y^3 - x^3}{xy^2}$$

(b) $f(x, y) = e^{xy^2}$

Using the Chain Rule, we have

$$f_x(x, y) = ye^{xy^2}$$

$$f_y(x, y) = 2xye^{xy^2}$$
(c) \( f(x, y) = y \ln(xy) \)

Using the Chain Rule, we have

\[
 f_x(x, y) = y \left( \frac{y}{xy} \right) = \frac{y}{x}
\]

Using the Product and Chain Rules, we have

\[
 f_y(x, y) = \ln(xy) + y \left( \frac{x}{xy} \right) = \ln(xy) + 1
\]

7. Find all first-order partial derivatives of each function.

(a) \( f(x, y, z) = x^3y^2z + \frac{x}{yz} \)

For simplicity, we rewrite the function as

\[
 f(x, y, z) = x^3y^2z + xy^{-1}z^{-1}.
\]

Therefore, we have

\[
 f_x(x, y, z) = 3x^2y^2z + \frac{1}{yz}
\]

\[
 f_y(x, y, z) = 2x^3yz - \frac{x}{y^2z}
\]

\[
 f_z(x, y, z) = x^3y^2 - \frac{x}{yz^2}
\]

(b) \( f(x, y, z) = e^{y \cos z} \sin x \)

Treating \( y \) and \( z \) as constants, we have

\[
 f_x(x, y, z) = e^{y \cos z} \cos x.
\]

Using the Chain Rule, we have

\[
 f_y(x, y, z) = \cos z e^{y \cos z} \sin x
\]

\[
 f_z(x, y, z) = -y \sin z e^{y \cos z} \sin x
\]
8. Find all second-order partial derivatives of each function.

(a) \( f(x, y) = x^2 y + x\sqrt{y} \)

The first-order partial derivatives of \( f \) are

\[
\begin{align*}
  f_x(x, y) &= 2xy + \sqrt{y} \\
  f_y(x, y) &= x^2 + \frac{x}{2\sqrt{y}}
\end{align*}
\]

Therefore, we find that

\[
\begin{align*}
  f_{xx}(x, y) &= 2y \\
  f_{xy}(x, y) &= 2x + \frac{1}{2\sqrt{y}} \\
  f_{yx}(x, y) &= 2x + \frac{1}{2\sqrt{y}} \\
  f_{yy}(x, y) &= -\frac{x}{4y^{3/2}}
\end{align*}
\]

(b) \( f(x, y) = \sin(x + y) + \cos(x - y) \)

The first-order partial derivatives of \( f \) are

\[
\begin{align*}
  f_x(x, y) &= \cos(x + y) - \sin(x - y) \\
  f_y(x, y) &= \cos(x + y) + \sin(x - y)
\end{align*}
\]

Therefore, we find that

\[
\begin{align*}
  f_{xx}(x, y) &= -\sin(x + y) - \cos(x - y) \\
  f_{xy}(x, y) &= -\sin(x + y) + \cos(x - y) \\
  f_{yx}(x, y) &= -\sin(x + y) + \cos(x - y) \\
  f_{yy}(x, y) &= -\sin(x + y) - \cos(x - y)
\end{align*}
\]
9. Find an equation of the tangent plane to \( f(x, y) = \sqrt{4 - x^2 - 2y^2} \) at \((1, -1)\).

At the point \((x_0, y_0) = (1, -1)\), we have \( z_0 = f(1, -1) = 1 \). Using the Chain Rule, we have

\[
\begin{align*}
    f_x(x, y) &= \frac{-x}{\sqrt{4 - x^2 - 2y^2}} \\
    f_y(x, y) &= \frac{-2y}{\sqrt{4 - x^2 - 2y^2}}
\end{align*}
\]

At the point \((1, -1)\), the partial derivatives are \( f_x(1, -1) = -1 \) and \( f_y(1, -1) = 2 \). Therefore, an equation of the tangent plane is

\[
\begin{align*}
    z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
    z - 1 &= -1(x - 1) + 2(y + 1) \\
    z - 1 &= -x + 2y + 3 \\
    x - 2y + z &= 4
\end{align*}
\]

10. Find the linearization of \( f(x, y) = y \cos(x - y) \) at \((2, 2)\) and use it to approximate \( f(1.9, 2.1) \). Check the accuracy of your approximation using a calculator.

Using the Product and Chain Rules, we have

\[
\begin{align*}
    f_x(x, y) &= -y \sin(x - y) \\
    f_y(x, y) &= \cos(x - y) + y \sin(x - y)
\end{align*}
\]

At the point \((2, 2)\), the partial derivatives are \( f_x(2, 2) = 0 \) and \( f_y(2, 2) = 1 \). Moreover, \( z = f(2, 2) = 2 \). Therefore, the linearization of \( f \) at \((2, 2)\) is

\[
\begin{align*}
    L(x, y) &= z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
    L(x, y) &= 2 + 0(x - 2) + 1(y - 2) \\
    L(x, y) &= y
\end{align*}
\]

To approximate \( f(1.9, 2.1) \), we use the linearization

\[
f(1.9, 2.1) \approx L(1.9, 2.1) = 2.1.
\]

Using a calculator, we find that

\[
f(1.9, 2.1) = 2.1 \cos(-0.2) \approx 2.06.
\]
11. Find the linearization of \( \mathbf{F}(x, y) = (e^{x^2-y^2}, \ln(2x-y^2))^T \) at \((1, 1)\).

First, \( \mathbf{F}(1, 1) = (1, 0)^T \). The Jacobian matrix for \( \mathbf{F}(x, y) \) is

\[
J(x, y) = \begin{bmatrix}
\frac{2x e^{x^2-y^2}}{2} & -2ye^{x^2-y^2} \\
\frac{2}{2x-y^2} & -\frac{2y}{2x-y^2}
\end{bmatrix}
\]

At \((1, 1)\), we have

\[
J(1, 1) = \begin{bmatrix} 2 & -2 \\
2 & -2 \end{bmatrix}
\]

Thus, the linearization of \( \mathbf{F}(x, y) \) at \((1, 1)\) is

\[
\mathbf{L}(x, y) = \begin{bmatrix} 1 \\
0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\
2 & -2 \end{bmatrix} \begin{bmatrix} x - 1 \\
y - 1 \end{bmatrix}
\]

12. Let \( f(x, y) = e^x \sin y \), where \( x(t) = t^2 \) and \( y(t) = 4t \). Find \( \frac{df}{dt} \) when \( t = 0 \).

Using the Chain Rule for functions of one parameter,

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}
\]

\[
= e^x \sin y(2t) + e^x \cos y(4)
\]

If \( t = 0 \), then \( x = y = 0 \) and

\[
\frac{df}{dt} = 0 + 4 = 4
\]
13. Compute the directional derivative of \( f(x, y) = \sqrt{xy - 2x^2} \) at the point \( P = (1, 6) \) in the direction of the point \( Q = (3, 1) \).

The vector which defines the direction is
\[
\vec{v} = \overrightarrow{PQ} = \langle 2, -5 \rangle
\]

A unit vector in the direction of \( \vec{v} \) is
\[
\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \left\langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}} \right\rangle
\]

The gradient vector of \( f \) is
\[
\nabla f(x, y) = \left\langle \frac{y - 4x}{2\sqrt{xy - 2x^2}}, \frac{x}{2\sqrt{xy - 2x^2}} \right\rangle
\]

At the point \( P = (1, 6) \),
\[
\nabla f(1, 6) = \left\langle \frac{1}{2}, \frac{1}{4} \right\rangle
\]

Therefore, the directional derivative of \( f \) at \( P \) in the direction of \( Q \) is
\[
D_{\vec{u}}f(1, 6) = \nabla f(1, 6) \cdot \vec{u} = \frac{1}{\sqrt{29}} - \frac{5}{4\sqrt{29}} = -\frac{1}{4\sqrt{29}}
\]

14. In what direction does \( f(x, y) = \sqrt{x^2 - y^2} \) increase most rapidly at \( (5, 3) \)? What is the largest increase?

The function increases most rapidly in the direction of the gradient vector:
\[
\nabla f(x, y) = \left\langle \frac{x}{\sqrt{x^2 - y^2}}, -\frac{y}{\sqrt{x^2 - y^2}} \right\rangle
\]

At the point \( (5, 3) \), we have
\[
\nabla f(5, 3) = \left\langle \frac{5}{4}, -\frac{3}{4} \right\rangle
\]

The largest increase is given by
\[
||\nabla f(5, 3)|| = \sqrt{\frac{25}{16} + \frac{9}{16}} = \sqrt{\frac{34}{16}} = \frac{\sqrt{34}}{4}
\]
15. Find a unit vector that is normal to the level curve of the function

\[ f(x, y) = x^2 + \frac{y^2}{4} \]

at (1, 2).

The gradient vector is normal to the level curve:

\[ \nabla f(x, y) = \left\langle 2x, \frac{y}{2} \right\rangle \]

At the point (1, 2), we have

\[ \nabla f(1, 2) = \langle 2, 1 \rangle \]

A unit vector in the direction of \( \nabla f(1, 2) \) is

\[ \vec{u} = \frac{\nabla f(1, 2)}{||\nabla f(1, 2)||} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle \]

16. Find all local extrema and saddle points of \( f(x, y) = -x^2 - y^2 + 6x + 8y - 21 \).

We first locate the critical points by setting the partial derivatives equal to zero:

\[ f_x = -2x + 6 = 0 \quad \text{and} \quad f_y = -2y + 8 = 0 \]

The only critical point is (3, 4).

Next we compute the second partial derivatives:

\[ f_{xx} = -2 \quad f_{xy} = 0 \quad f_{yy} = -2 \]

The discriminant is \( D(x, y) = 4 \). Since \( D(3, 4) = 4 > 0 \) and \( f_{xx}(3, 4) = -2 < 0 \), there is a local maximum at (3, 4). The local maximum value is \( f(3, 4) = 4 \). There are no local minima or saddle points.
17. Find all local extrema and saddle points of \( f(x, y) = 2x^4 + y^2 - 12xy \).

We first locate the critical points by setting the partial derivatives equal to zero:

\[
\begin{align*}
  f_x &= 8x^3 - 12y = 0 \\
  f_y &= 2y - 12x = 0
\end{align*}
\]

It follows from the second equation that \( y = 6x \). Substituting this into the first equation, we have

\[
\begin{align*}
  8x^3 - 12(6x) &= 0 \\
  8x^3 - 72x &= 0 \\
  8x(x^2 - 9) &= 0 \\
  x &= 0, \pm 3
\end{align*}
\]

The three critical points are \((0, 0)\), \((3, 18)\), and \((-3, -18)\).

Next we compute the second partial derivatives

\[
\begin{align*}
  f_{xx} &= 24x^2 \\
  f_{xy} &= -12 \\
  f_{yy} &= 2
\end{align*}
\]

The discriminant is

\[
D(x, y) = 48x^2 - 144
\]

Since \( D(0, 0) = -144 < 0 \), \((0, 0, 0)\) is a saddle point. Since \( D(\pm 3, \pm 18) = 288 > 0 \) and \( f_{xx}(\pm 3, \pm 18) = 216 > 0 \), there are local minima at \((3, 18)\) and \((-3, -18)\). The local minimum value is \( f(\pm 3, \pm 18) = -162 \). There are no local maxima.
18. Find the absolute extrema of \( f(x, y) = x^2 - y^2 + 4x + y \) on the rectangular region

\[ R = \{(x, y) | -4 \leq x \leq 0, 0 \leq y \leq 1\} \]

Since \( f \) is a polynomial, it is continuous on the closed, bounded rectangular region \( R \). Thus, \( f \) attains both an absolute maximum and an absolute minimum on \( R \).

We first find the critical points by setting the partial derivatives equal to zero:

\[
\begin{align*}
    f_x &= 2x + 4 = 0 \quad \text{and} \quad f_y = -2y + 1 = 0.
\end{align*}
\]

The only critical point is \((-2, \frac{1}{2})\) and the value of \( f \) there is \( f(-2, \frac{1}{2}) = -\frac{15}{4} \).

Next we look at the values of \( f \) on the boundary of \( R \) which consists of four edges.

On \( E_1 \), we have \( y = 0 \) and

\[
    f(x, 0) = x^2 + 4x \quad -4 \leq x \leq 0.
\]

To find the critical values of \( f \) on \( E_1 \), set

\[
    f'(x) = 2x + 4 = 0
\]

So \((-2, 0)\) is a critical point and \( f(-2, 0) = -4 \).

On \( E_2 \), we have \( x = 0 \) and

\[
    f(0, y) = -y^2 + y \quad 0 \leq y \leq 1
\]

To find the critical values of \( f \) on \( E_2 \), set

\[
    f'(y) = -2y + 1 = 0
\]

So \((0, \frac{1}{2})\) is a critical point and \( f(0, \frac{1}{2}) = \frac{1}{4} \).

On \( E_3 \), we have \( y = 1 \) and

\[
    f(x, 1) = x^2 + 4x \quad -4 \leq x \leq 0
\]

To find the critical values of \( f \) on \( E_3 \), set

\[
    f'(x) = 2x + 4 = 0
\]

So \((-2, 1)\) is a critical point and \( f(-2, 1) = -4 \).

On \( E_4 \), we have \( x = -4 \) and

\[
    f(-4, y) = -y^2 + y \quad 0 \leq y \leq 1
\]

To find the critical values of \( f \) on \( E_4 \), set

\[
    f'(y) = -2y + 1 = 0
\]

So \((-4, \frac{1}{2})\) is a critical point and \( f(-4, \frac{1}{2}) = \frac{1}{4} \).

Evaluating the function at each of the vertices on the boundary, we have \( f(0, 0) = 0 \), \( f(-4, 0) = 0 \), \( f(0, 1) = 0 \), and \( f(-4, 1) = 0 \). Therefore, the absolute maximum value is \( 1/4 \) and the absolute minimum value is \(-4 \).
19. Find the absolute extrema of \( f(x, y) = x^2 + 3y^2 + 2y \) on the disk

\[ D = \{(x, y)|x^2 + y^2 \leq 1\} \]

Since \( f \) is a polynomial, it is continuous on the closed, bounded disk \( D \). Thus, \( f \) attains both an absolute maximum and an absolute minimum on \( D \).

We first find the critical points by setting the partial derivatives equal to zero:

\[
\begin{align*}
  f_x &= 2x = 0 \quad \text{and} \quad f_y = 6y + 2 = 0 \\
\end{align*}
\]

The only critical point in \( D \) is \((0, -1/3)\).

Next we calculate the extreme values of \( f \) on the boundary of \( D \). That is, on the circle \( x^2 + y^2 = 1 \). On this circle, \( x^2 = 1 - y^2 \) where \(-1 \leq y \leq 1\). Therefore, we have

\[
\begin{align*}
  f(1 - y^2, y) &= (1 - y^2) + 3y^2 + 2y \\
  f(1 - y^2, y) &= 2y^2 + 2y + 1 \\
\end{align*}
\]

To find the critical values of \( f \) on the boundary of \( D \), let

\[
f'(1 - y^2, y) = 4y + 2 = 0.
\]

It follows that \( y = -1/2 \) and \( x = \sqrt{1 - y^2} = \sqrt{3}/2 \). Thus, a critical point on the boundary of \( D \) is \((\sqrt{3}/2, -1/2)\). The endpoints \((0, -1)\) and \((0, 1)\) are points of interest. Evaluating \( f \) at each of these points, we have

\[
\begin{align*}
  f\left(0, -\frac{1}{3}\right) &= -\frac{1}{3} \\
  f\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) &= \frac{1}{2} \\
  f(0, 1) &= 5 \\
  f(0, -1) &= 1 \\
\end{align*}
\]

The absolute maximum value is 5 and the absolute minimum value is \(-1/3\).
20. Consider the linear system
\[\begin{align*}
x_1(t+1) &= -0.4x_1(t) + 0.2x_2(t) \\
x_2(t+1) &= -0.3x_1(t) + 0.1x_2(t)
\end{align*}\]
Determine the stability of \(\bm{x} = \bm{0}\).

Let \(\bm{x}(t) = \langle x_1(t), x_2(t) \rangle^T\). The system can be written in matrix form as
\[
\bm{x}(t+1) = \begin{bmatrix} -0.4 & 0.2 \\ -0.3 & 0.1 \end{bmatrix} \bm{x}(t)
\]
To determine the stability of \(\bm{x} = \bm{0}\), we consider the eigenvalues of the coefficient matrix \(\bm{A}\). Let
\[
\begin{vmatrix} -0.4 - \lambda & 0.2 \\ -0.3 & 0.1 - \lambda \end{vmatrix} = \lambda^2 + 0.3\lambda + 0.02 = (\lambda + 0.1)(\lambda + 0.2) = 0
\]
The eigenvalues are \(\lambda_1 = -0.1\) and \(\lambda_2 = -0.2\). Since \(|\lambda_1| < 1\) and \(|\lambda_2| < 1\), the equilibrium \(\bm{x} = \bm{0}\) is locally stable.

21. Consider the linear system
\[\begin{align*}
x_1(t+1) &= -0.2x_1(t) - 0.4x_2(t) \\
x_2(t+1) &= 0.6x_1(t) + 0.1x_2(t)
\end{align*}\]
Determine the stability of \(\bm{x} = \bm{0}\).

Let \(\bm{x}(t) = \langle x_1(t), x_2(t) \rangle^T\). The system can be written in matrix form as
\[
\bm{x}(t+1) = \begin{bmatrix} -0.2 & -0.4 \\ 0.6 & 0.1 \end{bmatrix} \bm{x}(t)
\]
To determine the stability of \(\bm{x} = \bm{0}\), we consider the eigenvalues of the coefficient matrix \(\bm{A}\). Let
\[
\begin{vmatrix} -0.2 - \lambda & -0.4 \\ 0.6 & 0.1 - \lambda \end{vmatrix} = (-0.2 - \lambda)(0.1 - \lambda) + 0.24 = \lambda^2 + 0.1\lambda + 0.22 = 0
\]
The eigenvalues are given by
\[
\lambda = \frac{-0.1 \pm \sqrt{(0.1)^2 - 4(0.22)}}{2}
\]
Thus, the eigenvalues are complex conjugates. Since \(\det(\bm{A}) = 0.22 < 1\), it follows that \(|\lambda_1| < 1\) and \(|\lambda_2| < 1\). Thus, the equilibrium \(\bm{x} = \bm{0}\) is locally stable.
22. Find all nonnegative equilibria of

\[
\begin{align*}
    x_1(t+1) &= 2x_1(t)[1-x_1(t)] \\
    x_2(t+1) &= x_1(t)[1-x_2(t)]
\end{align*}
\]

and determine their stability.

To find the equilibria, let

\[
\begin{align*}
    x_1 &= 2x_1(1-x_1) \\
    x_2 &= x_1(1-x_2)
\end{align*}
\]

Simplifying the first equation, we have

\[
\begin{align*}
    2x_1^2 - x_1 &= 0 \\
    x_1(2x_2 - 1) &= 0
\end{align*}
\]

Thus, \(x_1 = 0\) or \(x_1 = 1/2\). If \(x_1 = 0\), then the second equation gives \(x_2 = 0\), and if \(x_1 = 1/2\), then the second equation gives \(x_2 = 1/3\). Therefore, the equilibria are \((0,0)\) and \((1/2,1/3)\).

The Jacobian matrix is

\[
J(x_1, x_2) = \begin{bmatrix}
    2 - 4x_1 & 0 \\
    1 - x_2 & -x_1
\end{bmatrix}
\]

At \((0,0)\), we have

\[
J(0,0) = \begin{bmatrix}
    2 & 0 \\
    1 & 0
\end{bmatrix}
\]

The eigenvalues are \(\lambda_1 = 2\) and \(\lambda_2 = 0\). Since \(|\lambda_1| > 1\), it follows that \((0,0)\) is unstable.

At \((1/2,1/3)\), we have

\[
J \left( \frac{1}{2}, \frac{1}{3} \right) = \begin{bmatrix}
    0 & 0 \\
    \frac{2}{3} & -\frac{1}{2}
\end{bmatrix}
\]

The eigenvalues are \(\lambda_1 = 0\) and \(\lambda_2 = -1/2\). Since \(|\lambda_1| < 1\) and \(|\lambda_2| < 1\), it follows that \((1/2,1/3)\) is locally stable.
23. Find all biologically-relevant equilibria of the Nicholson-Bailey model

\[
N_{t+1} = 4N_t e^{-0.1P_t} \\
P_{t+1} = N_t \left[ 1 - e^{-0.1P_t} \right]
\]

Determine the stability of the extinction equilibrium.

To find the equilibria, let

\[
N = 4Ne^{-0.1P} \\
P = N \left[ 1 - e^{-0.1P} \right]
\]

Using the first equation, we obtain

\[
N - 4Ne^{-0.1P} = 0 \\
N \left( 1 - 4e^{-0.1P} \right) = 0
\]

So either \( N = 0 \) or, if \( N \neq 0 \), then

\[
1 - 4e^{-0.1P} = 0 \\
\frac{1}{4} = e^{-0.1P} \\
4 = e^{0.1P} \\
\ln(4) = 0.1P \\
P = 10 \ln(4)
\]

If \( N = 0 \), then the second equation yields \( P = 0 \). Thus, the extinction equilibrium is \((N^*, P^*) = (0, 0)\). If \( P = 10 \ln(4) \), then the second equation yields

\[
10 \ln(4) = \frac{3}{4} N \\
N = \frac{40}{3} \ln(4)
\]

Thus, the coexistence equilibrium is

\[
(N^*, P^*) = \left( \frac{40}{3} \ln(4), 10 \ln(4) \right)
\]

The Jacobian matrix is

\[
J(N, P) = \begin{bmatrix}
4e^{-0.1P} & -0.4Ne^{-0.1P} \\
1 - e^{-0.1P} & 0.1Ne^{-0.1P}
\end{bmatrix}
\]

At the extinction equilibrium \((0, 0)\), we have

\[
J(0, 0) = \begin{bmatrix}
4 & 0 \\
0 & 0
\end{bmatrix}
\]

The eigenvalues are \( \lambda_1 = 4 \) and \( \lambda_2 = 0 \). Since \(|\lambda_1| = 4 > 1\), the extinction equilibrium is unstable.