Section 14.7: Surface Integrals

Suppose $f$ is a function of three variables defined on a surface $S$. Partition $S$ into patches $S_{ij}$ with area $\Delta S_{ij}$. Evaluate $f$ at a point $P_{ij}^*$ in each patch and form the sum

$$
\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}.
$$

Let $||P||$ denote the norm of the partition $P$ which is the area of the largest patch.

Definition: The surface integral of $f$ over the surface $S$ is

$$
\iint_{S} f(x, y, z) dS = \lim_{||P|| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}
$$

provided this limit exists.

Theorem: (Formula for Surface Integrals)

Suppose $f$ is a function of three variables defined on a surface $S$ with equation $z = g(x, y)$ for $(x, y) \in D$. Then

$$
\iint_{S} f(x, y, z) dS = \iint_{D} f(x, y, g(x, y)) \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} dA.
$$

Example: Evaluate the surface integral $\iint_{S} x dS$, where $S$ is the part of the plane $2x+2y+z=6$ that lies in the first octant.

The projection of $S$ onto the $xy$-plane is the region

$$
D = \{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 3 - x\}.
$$

Then $z_x = -2$, $z_y = -2$, and

$$
\iint_{S} y dS = \int_{0}^{3} \int_{0}^{3-x} y \sqrt{(-2)^2 + (-2)^2 + 1} dy dx
$$

$$
= \frac{3}{2} \int_{0}^{3} y^2 \bigg|_{0}^{3-x} dx
$$

$$
= \frac{3}{2} \int_{0}^{3} (3-x)^2 dx
$$

$$
= \frac{1}{2} (3-x)^3 \bigg|_{0}^{3}
$$

$$
= \frac{27}{2}.
$$
Theorem: (Center of Mass of a Surface)
If a thin sheet has the shape of a surface \(S\) and the density at \((x, y, z)\) is \(\rho(x, y, z)\), then the total mass of the sheet is
\[
m = \iint_S \rho(x, y, z)\,dS
\]
and the center of mass is \((\bar{x}, \bar{y}, \bar{z})\), where
\[
\bar{x} = \frac{1}{m} \iint_S x\rho(x, y, z)\,dS \quad \bar{y} = \frac{1}{m} \iint_S y\rho(x, y, z)\,dS \quad \bar{z} = \frac{1}{m} \iint_S z\rho(x, y, z)\,dS.
\]

Example: Find the mass of a thin funnel in the shape of a cone \(z = \sqrt{x^2 + y^2}\), \(1 \leq z \leq 4\) if its density function is \(\rho(x, y, z) = 10 - z\).

First,
\[
\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.
\]

The mass of the funnel is
\[
m = \iint_S (10 - z)\,dS = \sqrt{2} \iint_D (10 - \sqrt{x^2 + y^2})\,dA
\]
\[
= \sqrt{2} \int_0^{2\pi} \int_1^4 (10 - r) r\,dr\,d\theta
\]
\[
= 2\sqrt{2}\pi \int_1^4 (10r - r^2)\,dr
\]
\[
= 2\sqrt{2}\pi \left(5r^2 - \frac{1}{3}r^3\right) \bigg|_1^4
\]
\[
= 108\sqrt{2}\pi.
\]

Theorem: (Surface Integral for a Parametric Surface)
Suppose \(f\) is a function of three variables defined on a surface \(S\). If \(S\) has a vector equation \(\vec{R}(u, v) = (x(u, v), y(u, v), z(u, v))\) for \((u, v) \in D\), then
\[
\iint_S f(x, y, z)\,dS = \iint_D f(\vec{R}(u, v))||\vec{R}_u \times \vec{R}_v||\,dA.
\]
Example: Evaluate the surface integral $\int\int_S yz\,dS$, where $S$ is the helicoid with vector equation $\vec{R}(u,v) = \langle u \cos v, u \sin v, v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq \pi$.

The partial derivatives are

$$\vec{R}_u = \langle \cos v, \sin v, 0 \rangle \quad \text{and} \quad \vec{R}_v = \langle -u \sin v, u \cos v, 1 \rangle.$$ 

Then

$$\vec{R}_u \times \vec{R}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle.$$ 

Thus

$$||\vec{R}_u \times \vec{R}_v|| = \sqrt{1 + u^2}.$$ 

Therefore,

$$\int\int_S yz\,dS = \int_0^\pi \int_0^1 uv \sin v \sqrt{u^2 + 1} \,du \,dv = \left( \int_0^1 u \sqrt{u^2 + 1} \,du \right) \left( \int_0^\pi v \sin v \,dv \right) = \left( \frac{1}{3} (u^2 + 1)^{3/2} \right)^1_0 (\cos v + \sin v)^\pi_0 \right) \right) = \frac{\pi}{3} (2\sqrt{2} - 1).$$

Definition: A surface $S$ is called orientable if it has a normal vector field $\vec{N}$ that varies continuously over $S$. If $S$ is orientable, there are two choices for $\vec{N}$ that provide $S$ with an orientation. For the positive orientation, the normal vectors point outward and for the negative orientation, the normal vectors point inward.

Note: Examples of non-orientable surfaces are the Möbius strip or Klein bottle.

Definition: If $\vec{F}$ is a continuous vector field defined on an orientable surface $S$ with unit normal vector $\vec{N}$, then the surface integral of $\vec{F}$ over $S$ is

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \vec{F} \cdot \vec{N} \,dS.$$ 

This integral is also called the flux of $\vec{F}$ across $S$. 
Theorem: (Formula for Flux)
If \( \vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle \) and \( S \) is given by \( z = g(x, y) \) for \( (x, y) \in D \), then
\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA,
\]
where \( \vec{N} \) is the upward unit normal. For the downward unit normal, multiply by \(-1\).

Example: Find the flux of \( \vec{F}(x, y, z) = \langle e^y, ye^x, x^2y \rangle \) across the part of the paraboloid \( z = \frac{x^2}{4} + \frac{y^2}{4} \) that lies above the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and has upward orientation.

Applying the previous theorem, the flux of \( \vec{F} \) across \( S \) is
\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left( -e^y(2x) - ye^x(2y) + x^2y \right) dA
\]
\[
= \int_0^1 \int_0^1 (x^2y - 2xe^y - 2y^2e^x) dy dx
\]
\[
= \int_0^1 \left( \frac{1}{2}x^2y^2 - 2xe^y - \frac{2}{3}y^3e^x \right) dx
\]
\[
= \left[ \frac{1}{2}x^2 + 2(1 - e)x - \frac{2}{3}e^x \right]_0^1
\]
\[
= \frac{1}{6}x^3 + (1 - e)x^2 - \frac{2}{3}e^x \bigg|_0^1
\]
\[
= \frac{1}{6} + (1 - e) - \frac{2}{3}e + \frac{2}{3}
\]
\[
= \frac{11}{6} - 10e.
\]

Note: If \( S \) is a piecewise-smooth surface, then the surface integral of \( \vec{F} \) over \( S \) is defined by
\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \cdots + \iint_{S_n} \vec{F} \cdot d\vec{S}.
\]

Example: Find the outward flux of \( \vec{F}(x, y, z) = \langle x, y, z \rangle \) across the boundary of the region enclosed by the paraboloid \( z = 4 - x^2 - y^2 \) and the plane \( z = 0 \).

The surface is given by \( S = S_1 \cup S_2 \), where \( S_1 \) is the paraboloid \( z = 4 - x^2 - y^2 \) and \( S_2 \) is the disk \( x^2 + y^2 \leq 4 \) in the \( xy \)-plane. Since \( S \) has the positive orientation, \( S_1 \) is oriented upward and \( S_2 \) is oriented downward. On \( S_1 \),
\[
\vec{F}(x, y, z) = \langle x, y, z \rangle = \langle x, y, 4 - x^2 - y^2 \rangle
\]
and 
\[ \frac{\partial g}{\partial x} = -2x, \quad \frac{\partial g}{\partial y} = -2y. \]

Then
\[
\int \int_S \vec{F} \cdot d\vec{S} = \int \int_D (2x^2 + 2y^2 + 4 - x^2 - y^2) dA \\
= \int \int_D (x^2 + y^2 + 4) dA \\
= \int_0^{2\pi} \int_0^2 (r^2 + 4) r dr d\theta \\
= 2\pi \int_0^2 (r^3 + 4r) dr \\
= 2\pi \left( \frac{1}{4} r^4 + 2r^2 \right) \bigg|_0^2 \\
= 24\pi.
\]

Since the disk \( S_2 \) is oriented downward, its normal vector is \( \vec{N} = (0, 0, -1) \) and
\[
\int \int_{S_2} \vec{F} \cdot \vec{N} dS = \int \int_D (-z) dA = \int \int_D 0 dA = 0
\]
since \( z = 0 \) on \( S_2 \). Thus,
\[
\text{Flux} = \int \int_S \vec{F} \cdot d\vec{S} = \int \int_{S_1} \vec{F} \cdot d\vec{S} + \int \int_{S_2} \vec{F} \cdot d\vec{S} = 24\pi.
\]

**Theorem: (Flux Integrals for Parametric Surfaces)**

If \( S \) is a parametric surface defined by a vector function \( \vec{R}(u, v) \) for \((u, v) \in D\), then
\[
\int \int_S \vec{F} \cdot d\vec{S} = \int \int_D \vec{F}(\vec{R}(u, v)) \cdot (\vec{R}_u \times \vec{R}_v) dA.
\]

**Example:** Find the outward flux of \( \vec{F}(x, y, z) = (z, y, x) \) across the sphere \( x^2 + y^2 + z^2 = 1 \).

Using spherical coordinates, the sphere can be parameterized by
\[ \vec{R}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \]
for \( 0 \leq \phi \leq \pi \) and \( 0 \leq \theta \leq 2\pi \). Then
\[ \vec{R}_\phi = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi) \]
\[ \vec{R}_\theta = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0) \]
\[ \vec{F}(\phi, \theta) = (\cos \phi, \sin \phi \sin \theta, \sin \phi \cos \theta). \]
Thus,
\[
\vec{R}_\phi \times \vec{R}_\theta = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\
-\sin \phi \sin \theta & \sin \phi \cos \theta & 0
\end{vmatrix} = \langle \sin^2 \phi \cos \theta, \sin^2 \phi \sin \theta, \sin \phi \cos \phi \rangle.
\]

Then
\[
\vec{F}(\phi, \theta) \cdot (\vec{R}_\phi \times \vec{R}_\theta) = 2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta.
\]

Therefore,
\[
\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^\pi \left(2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta\right)d\phi d\theta
\]
\[
= 2 \int_0^\pi \sin^2 \phi \cos \phi d\phi \int_0^{2\pi} \cos \theta d\theta + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta
\]
\[
= \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta
\]
\[
= \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi \int_0^{2\pi} 1 - \cos 2\theta \frac{1}{2} d\theta
\]
\[
= \left(-\cos \phi + \frac{1}{3} \cos^3 \phi\right) \left(\frac{\theta}{2} - \frac{1}{4} \sin 2\theta\right)_{\theta=0}^{\theta=2\pi}
\]
\[
= \frac{4\pi}{3}.
\]

**Definition:** Suppose the temperature at a point \((x, y, z)\) in a body is \(T(x, y, z)\). Then the **heat flow** is defined as the vector field

\[
\vec{F} = -K \nabla T,
\]
where \(K\) is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across a surface \(S\) in the body is given by the flux integral.

**Example:** The temperature at the point \((x, y, z)\) in a substance with conductivity \(K = 6.5\) is \(T(x, y, z) = 2(x^2 + y^2)\). Find the rate of heat flow inward across the cylindrical surface \(x^2 + y^2 = 6\) for \(0 \leq z \leq 4\).

Using cylindrical coordinates, the surface \(S\) can be parameterized as
\[
\vec{R}(\theta, z) = \langle \sqrt{6} \cos \theta, \sqrt{6} \sin \theta, z \rangle
\]
for \(0 \leq \theta \leq 2\pi\) and \(0 \leq z \leq 4\). Then
\[
\vec{R}_\theta = \langle -\sqrt{6} \sin \theta, \sqrt{6} \cos \theta, 0 \rangle \quad \text{and} \quad \vec{R}_z = \langle 0, 0, 1 \rangle.
\]
Thus,
\[
\vec{R}_\theta \times \vec{R}_z = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
-\sqrt{6} \sin \theta & \sqrt{6} \cos \theta & 0 \\
0 & 0 & 1 \\
\end{vmatrix} = (\sqrt{6} \cos \theta, \sqrt{6} \sin \theta, 0).
\]

The heat flow is
\[
\vec{F}(x, y, z) = -K \vec{\nabla} T = -6.5 \langle 4x, 4y, 0 \rangle = \langle -26x, -26y, 0 \rangle.
\]

Therefore,
\[
\vec{F}(\theta, z) = \langle -26\sqrt{6} \cos \theta, -26\sqrt{6} \sin \theta, 0 \rangle
\]

and
\[
\vec{F}(\theta, z) \cdot (\vec{R}_\theta \times \vec{R}_z) = -156(\cos^2 \theta + \sin^2 \theta) = -156.
\]

Thus, the rate of heat flow inward across the surface \( S \) is
\[
- \int \int_S \vec{F} \cdot d\vec{S} = - \int_0^{2\pi} \int_0^4 -156 \, dz \, d\theta = 1248\pi.
\]