Section 3.5: Nonhomogeneous Equations; Method of Undetermined Coefficients

Consider a second order nonhomogeneous linear equation

\[ y'' + p(t)y' + q(t)y = g(t), \]  

where \( p, q, \) and \( g \) are continuous functions on an open interval \( I. \) The equation

\[ y'' + p(t)y' + q(t)y = 0, \]  

in which \( g(t) = 0 \) is called the corresponding homogeneous equation.

**Theorem 3.5.2:** The general solution of the nonhomogeneous equation

\[ y'' + p(t)y' + q(t)y = g(t), \]

can be written in the form

\[ y(t) = c_1y_1(t) + c_2y_2(t) + Y(t), \]

where \( y_1 \) and \( y_2 \) are a fundamental set of solutions of the corresponding homogeneous ODE, \( c_1 \) and \( c_2 \) are arbitrary constants, and \( Y \) is some solution of the nonhomogeneous equation.

**Method of Undetermined Coefficients:**
To find the general solution of the nonhomogeneous equation

\[ y'' + p(t)y' + q(t)y = g(t), \]

1. Find the general solution \( c_1y_1(t) + c_2y_2(t) \) of the corresponding homogeneous equation. This solution is frequently called the **homogeneous solution** or **complementary solution** and denoted by \( y_h(t) \) or \( y_c(t) \).

2. Find some solution \( Y(t) \) of the nonhomogeneous equation. This solution is often called a **particular solution** and denoted by \( y_p(t) \). To find a particular solution, we must make an initial assumption about the form of \( y_p(t) \).

3. The general solution is given by the sum \( y_h(t) + y_p(t) \).
Example: Find the general solution of

\[ y'' + 3y' + 2y = 12t^2. \]
Example: Find the general solution of

\[ y'' - y' - 2y = 20e^{3t}. \]
Example: Find the general solution of

\[ y'' + 2y' + 5y = 3 \sin 2t. \]
Example: Find the general solution of

\[ y'' - 3y' - 4y = 2e^{-t}. \]
How to Choose the Form of a Particular Solution

Consider the nonhomogeneous equation

\[ ay'' + by' + cy = g(t). \]

1. If \( g(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n \), then use a particular solution of the form

\[ y_p(t) = t^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_n), \]

where \( s \) is the number of times 0 is a root of the characteristic equation.

2. If \( g(t) = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n) e^{rt} \), then use a particular solution of the form

\[ y_p(t) = t^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{rt}. \]

   (a) If \( r \) is not a root of the characteristic equation, then \( s = 0 \).

   (b) If \( r \) is a simple root of the characteristic equation, then \( s = 1 \).

   (c) If \( r \) is a double root of the characteristic equation, then \( s = 2 \).

3. If \( g(t) = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n) e^{\alpha t} \sin(\beta t) \) or \( g(t) = (a_0 t^n + a_1 t^{n-1} + \cdots + a_n) e^{\alpha t} \cos(\beta t) \), then use a particular solution of the form

\[ y_p(t) = t^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) e^{\alpha t} \sin(\beta t) + t^s (B_0 t^n + B_1 t^{n-1} + \cdots + B_n) e^{\alpha t} \cos(\beta t). \]

   (a) If \( \alpha \pm \beta i \) are not the roots of the characteristic equation, then \( s = 0 \).

   (b) If \( \alpha \pm \beta i \) are the roots of the characteristic equation, then \( s = 1 \).

Example: Find the form of a particular solution for the differential equation

\[ y'' + 2y' - 3y = g(t), \]

where \( g(t) \) is given by the following functions. (Do not evaluate the coefficients.)

\[
\begin{align*}
(a) \quad & 7 \cos 3t & (d) \quad & 5e^{-3t} \\
(b) \quad & 2te^t \sin t & (e) \quad & 3te^t \\
(c) \quad & t^2 \cos \pi t & (f) \quad & t^2 e^t
\end{align*}
\]