

Answer Key

MATH 251 Test II October 26, 2004 Name _____

Section _____

1. _____ (25%)
2. _____ (15%)
3. _____ (20%)
4. _____ (20%)
5. _____ (20%)
Total _____ (100%)

There are a total of five problems. No calculators are allowed.

1. A function $u = f(x, y)$ is said to be homogeneous of degree n for some $n > 0$ if $f(tx, ty) = t^n f(x, y)$.

(a) Show that

$$xf_x(x, y) + yf_y(x, y) = nf(x, y). \quad (5\%)$$

(b) Show that

$$x^2 f_{xx}(x, y) + y^2 f_{yy}(x, y) + 2xy f_{xy}(x, y) = n(n-1)f(x, y). \quad (5\%)$$

(c) Show that $f_y(tx, ty) = t^{n-1} f_y(x, y)$ for all $t > 0$. (5%)

(d) Show that $xf_x(2x, 2y) + yf_y(2x, 2y) = n \cdot 2^{n-1} f(x, y)$. (5%)

(e) Let $n = 5/2$. Give a concrete example of a function $f(x, y)$ that is homogeneous of degree $5/2$. (5%)

Solution :

(a) From $f(tx, ty) = t^n f(x, y)$, take $\frac{d}{dt}$:

$$\frac{d}{dt} f(tx, ty) = \frac{d}{dt} [t^n f(x, y)] = nt^{n-1} f(x, y).$$

Apply the chain rule to the left hand side above:

$$(*) \quad \frac{d}{dt} f(tx, ty) = \frac{\partial f(tx, ty)}{\partial x} \frac{d(tx)}{dt} + \frac{\partial f(tx, ty)}{\partial y} \frac{d(ty)}{dt} = x \frac{\partial f(tx, ty)}{\partial x} + y \frac{\partial f(tx, ty)}{\partial y} = nt^{n-1} f(x, y).$$

Now set $t=1$ above. We obtain

$$xf_x(x, y) + yf_y(x, y) = nf(x, y).$$

(b) Take $\frac{d}{dt}$ of both sides of $\textcircled{*}$ above:

$$\frac{d^2}{dt^2} f(tx, ty) = \frac{d}{dt} \left[x \frac{\partial f(tx, ty)}{\partial x} + y \frac{\partial f(tx, ty)}{\partial y} \right] = \frac{d}{dt} (nt^{n-1} f(x, y)) = n(n-1)t^{n-2} f(x, y).$$

Apply the chain rule to the bracketed term above:

$$\begin{aligned} \frac{d}{dt} \left[x \frac{\partial f(tx, ty)}{\partial x} + y \frac{\partial f(tx, ty)}{\partial y} \right] &= x \frac{d}{dt} f_x(tx, ty) + y \frac{d}{dt} f_y(tx, ty) \\ &= x \left[f_{xx}(tx, ty) \frac{d(tx)}{dt} + f_{xy}(tx, ty) \frac{d(ty)}{dt} \right] + y \left[f_{yx}(tx, ty) \frac{d(tx)}{dt} + f_{yy}(tx, ty) \frac{d(ty)}{dt} \right] \\ &= x \left[f_{xx}(tx, ty)x + f_{xy}(tx, ty)y \right] + y \left[f_{yx}(tx, ty)x + f_{yy}(tx, ty)y \right] \\ &= x^2 f_{xx}(tx, ty) + 2xy f_{xy}(tx, ty) + y^2 f_{yy}(tx, ty) = n(n-1)t^{n-2} f(x, y). \end{aligned}$$

Again, set $t=1$; we obtain

$$x^2 f_{xx}(x, y) + 2xy f_{xy}(x, y) + y^2 f_{yy}(x, y) = n(n-1)f(x, y).$$

(c)

$$\frac{\partial}{\partial y} f(tx, ty) = f_y(tx, ty) \frac{\partial(ty)}{\partial y} = t f_y(tx, ty) = \frac{\partial}{\partial y} [t^n f(x, y)] = t^n f_y(x, y).$$

In $t f_y(tx, ty) = t^n f_y(x, y)$ above, divide both sides by t . We obtain $f_y(tx, ty) = t^{n-1} f_y(x, y)$.

(d) Substitute $t=2$ in equation $\textcircled{*}$ on the preceding page. We obtain

$$x f_x(2x, 2y) + y f_y(2x, 2y) = n \cdot 2^{n-1} f(x, y).$$

(e) For example,

$$f(x, y) = 3xy^{\frac{3}{2}} - 7x^{5/2}$$

is homogeneous of degree 5.

2. Given $f(x, y) = (x+1)^4 + 256(y-2)^4 + 16[x(y-2) + y] - 90$,

(a) find all the critical points of f ; (10%)

(b) determine the local extrema of f . (5%)

[Hint: Calculations will be simplified by setting $u = x+1, v = 4(y-2)$, and by writing $16[x(y-2) + y]$ in terms of u and v .]

Solution. (a) Set $u = x+1$ and $v = 4(y-2)$. Then

$$\begin{aligned} 16[x(y-2) + y] &= 16[x(y-2) + (y-2) + 2] \\ &= 16[(x+1)(y-2) + 2] = 4 \cdot [(x+1)] [4(y-2)] + 32 \\ &= 4uv + 32 \end{aligned}$$

and $f(x, y) = u^4 + v^4 + 4uv + 32 - 90 = u^4 + v^4 + 4uv - 58 \stackrel{\text{def.}}{=} F(u, v)$.

We find the extremal points of $f(x, y)$ by finding the extremal points of $F(u, v)$.

$$\begin{aligned} \frac{\partial F}{\partial u} = 4u^3 + 4v = 0 &\Rightarrow u^3 + v = 0 \\ \frac{\partial F}{\partial v} = 4v^3 + 4u = 0 &\Rightarrow v^3 + u = 0 \end{aligned} \Rightarrow u = -v^3$$

$$0 = u^3 + v = (-v^3)^3 + v = -v^9 + v = 0$$

$$0 = v^9 - v = (v^4 + 1)(v^2 + 1)(v + 1)(v - 1)v = 0$$

Thus, $\begin{cases} v = 0, & u = 0 \\ v = 1, & u = -1 \\ v = -1, & u = 1 \end{cases}$ are 3 critical points, giving $\begin{cases} x = -1, & y = 2, \\ x = -2, & y = 9/4, \\ x = 0, & y = 7/4. \end{cases}$ respectively.

(b) $f_{xx} = 4 \cdot 3 \cdot (x+1)^2 \geq 0$, $f_{xy} = 16$.
 $f_{yy} = 256 \cdot 4 \cdot 3 (y-2)^2$

(i) At $(x, y) = (-1, 2)$, $D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 16^2 = -256 < 0$. The test is indeterminate.

(ii) At $(x, y) = (-2, \frac{9}{4})$, $D = 12 \cdot (16 \cdot 4 \cdot 3) - 256 > 0$, $f_{xx} = 12 > 0$. So this pt. is a local min.

(iii) at $(x, y) = (0, \frac{7}{4})$, $D = 12 \cdot (16 \cdot 4 \cdot 3) - 256 > 0$, $f_{xx} = 12 > 0$. So this pt. is a local min.

3. (a) Find the maximum and minimum value of the function

$$f(x, y, z) = 4x + 2y + 3z$$

subject to the constraints

$$2x + 2y + z = 10, \quad 4(y+1)^2 + z^2 = 16. \quad (14\%)$$

- (b) What is the name of the surface $4(y+1)^2 + z^2 = 16$? (3%)

- (c) Give a sketch of the constraint set. (3%)

(a) Let $g(x, y, z) = 2x + 2y + z - 10$, $h(x, y, z) = 4(y+1)^2 + z^2 - 16$

Then $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$

$$\begin{cases} 4 = \lambda \cdot 2 + \mu \cdot 0, \\ 2 = \lambda \cdot 2 + \mu \cdot 8(y+1), \\ 3 = \lambda \cdot 1 + \mu \cdot 2z. \end{cases}$$

$$\lambda = 2; \quad \begin{cases} 2 = 4 + \mu \cdot 8(y+1) \\ 3 = 4 + \mu \cdot 2z \end{cases} \Rightarrow \begin{cases} y+1 = -\frac{1}{4} \frac{1}{\mu} \\ z = \frac{1}{2} \frac{1}{\mu} \end{cases}$$

$$16 = 4(y+1)^2 + z^2 = 4 \cdot \left(-\frac{1}{4} \frac{1}{\mu}\right)^2 + \left(-\frac{3}{2} \frac{1}{\mu}\right)^2 = \left(\frac{1}{4} + \frac{9}{4}\right) \frac{1}{\mu^2} = \frac{10}{\mu^2}$$

$$\mu = \pm (4\sqrt{2})^{-1} = \pm \frac{\sqrt{2}}{8}$$

$$(i) \mu = \frac{\sqrt{2}}{8} \Rightarrow \begin{cases} y = -1 - \frac{1}{4} \frac{8}{\sqrt{2}} = -1 - \sqrt{2}, & z = -\frac{1}{2} \frac{8}{\sqrt{2}} = -2\sqrt{2} \\ x = \frac{1}{2}(10 - 2y - z) = 5 - (-1 - \sqrt{2}) - \frac{1}{2}(-2\sqrt{2}) = 6 \end{cases} \left. \vphantom{\begin{matrix} y \\ z \\ x \end{matrix}} \right\} (x, y, z) = (6, -1 - \sqrt{2}, -2\sqrt{2})$$

$$f(x, y, z) = 4 \cdot 6 + 2(-1 - \sqrt{2}) + 3(-2\sqrt{2}) = 22 - 4\sqrt{2} \quad (\text{max})$$

$$(ii) \mu = -\frac{\sqrt{2}}{8} \Rightarrow \begin{cases} y = -1 + \frac{1}{4} \frac{8}{\sqrt{2}} = -1 + \sqrt{2}, & z = \frac{1}{2} \left(-\frac{8}{\sqrt{2}}\right) = -2\sqrt{2} \\ x = \frac{1}{2}(10 - 2y - z) = \frac{1}{2}(10) - (-1 + \sqrt{2}) - \frac{1}{2}(-2\sqrt{2}) = 6 \end{cases} \left. \vphantom{\begin{matrix} y \\ z \\ x \end{matrix}} \right\} (x, y, z) = (6, -1 + \sqrt{2}, -2\sqrt{2})$$

$$f(x, y, z) = 4 \cdot 6 + 2(-1 + \sqrt{2}) + 3(-2\sqrt{2}) = 22 - 4\sqrt{2} \quad (\text{min})$$

4. (a) Evaluate $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx$ for $a > 0, b \in \mathbb{R}$ and $c > 0$. (8%)

(b) Use part (a) above to determine $\int_0^{\infty} e^{-(ax^2+c)} dx$, for $a > 0$. (4%)

(c) Use part (b) above to evaluate $\int_0^{\infty} x^2 e^{-2x^2} dx$. (4%)

(d) In part (a), can we evaluate the integral if $c < 0$?

(a) Consider $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx \cdot \int_{-\infty}^{\infty} e^{-(ay^2+by+c)} dy = \int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2 + (\frac{b^2}{4a}-c)} dx \cdot \int_{-\infty}^{\infty} e^{-a(y+\frac{b}{2a})^2 + (\frac{b^2}{4a}-c)} dy$ (4%)

(by the change of variables $x+\frac{b}{2a} \rightarrow x, y+\frac{b}{2a} \rightarrow y$)
 $= \int_{-\infty}^{\infty} e^{-ax^2} dx \cdot e^{\frac{b^2}{4a}-c} \cdot \int_{-\infty}^{\infty} e^{-ay^2} dy \cdot e^{\frac{b^2}{4a}-c} = e^{2(\frac{b^2}{4a}-c)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy$

(use polar coordinates \Rightarrow) $= e^{2(\frac{b^2}{4a}-c)} \int_0^{\infty} \int_0^{2\pi} e^{-ar^2} r dr d\theta$
 $= e^{2(\frac{b^2}{4a}-c)} \cdot 2\pi \cdot \left[-\frac{1}{2a} e^{-ar^2} \Big|_{r=0}^{r=\infty} \right] = e^{2(\frac{b^2}{4a}-c)} \cdot \frac{2\pi}{2a} = e^{2(\frac{b^2}{4a}-c)} \frac{\pi}{a}$

Thus $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = \sqrt{e^{2(\frac{b^2}{4a}-c)} \frac{\pi}{a}} = e^{\frac{b^2}{4a}-c} \sqrt{\frac{\pi}{a}}$

(b) By symmetry of the integrand $e^{-(ax^2+c)}$ about $x=0$, we have
 $\int_0^{\infty} e^{-(ax^2+c)} dx = \frac{1}{2} e^{\frac{b^2}{4a}-c} \sqrt{\frac{\pi}{a}}$

(c) $\int_0^{\infty} x^2 e^{-ax^2} dx = \int_0^{\infty} \left(-\frac{1}{a} \frac{\partial}{\partial a} e^{-ax^2} \right) dx = -\frac{1}{a} \frac{\partial}{\partial a} \int_0^{\infty} e^{-ax^2} dx = -\frac{1}{a} \frac{\partial}{\partial a} \left[\frac{1}{2} \sqrt{\frac{\pi}{a}} \right]$
 $= \frac{1}{2a} \sqrt{\frac{\pi}{a^3}}$. Now use $a=2$, giving

$$\int_0^{\infty} x^2 e^{-2x^2} dx = \frac{1}{2(2)} \sqrt{\frac{\pi}{2^3}} = \frac{1}{8} \sqrt{\frac{\pi}{2}}$$

(d) c is only a constant and e^{-c} is a constant. So the integral $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx$ is finite regardless of the sign of c .

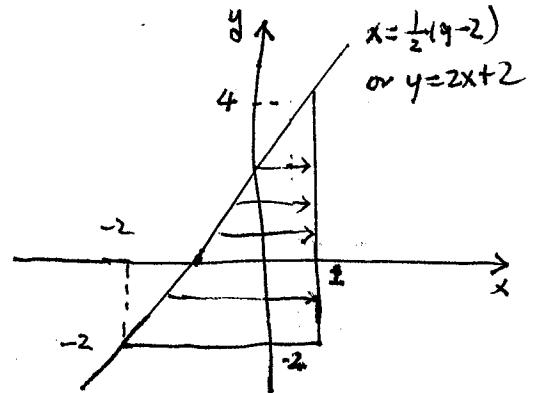
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5. Consider

$$\int_{-2}^4 \int_{\frac{1}{2}(y-2)}^1 \cos(3x^2 + 12x - 5) dx dy.$$

(a) Sketch the region where the integral is evaluated. (5%)

(b) Evaluate the integral by changing the order of integration. (15%)



$$\int_{-2}^4 \int_{\frac{1}{2}(y-2)}^1 \cos(3x^2 + 12x - 5) dx dy$$

$$= \int_{-2}^1 \int_{-2}^{2x+2} \cos(3x^2 + 12x - 5) dy dx$$

$$= \int_{-2}^1 \cos(3x^2 + 12x - 5) \left[y \Big|_{y=-2}^{y=2x+2} \right] dx$$

$$= \int_{-2}^1 \cos(3x^2 + 12x - 5) [(2x+2) - (-2)] dx$$

$$= \int_{-2}^1 (2x+4) \cos(3x^2 + 12x - 5) dx$$

$$= \frac{1}{3} \sin(3x^2 + 12x - 5) \Big|_{x=-2}^{x=1}$$

$$= \frac{1}{3} [\sin 10 - \sin(-17)] = \frac{1}{3} [\sin 10 + \sin 17]$$