**Math 251**

**Test I Fall 2005 (Sept. 30, 2005)**

Section 508 (8–9:15 am) 509 (9:35–10:50 am)

**Name** ____________________

**ID** ____________________

1. ______
2. ______
3. ______
4. ______

Total ______

There are a total of 4 problems. No calculators are allowed.

1. Let $n$ be a positive integer greater than 2, and

$$f(x_1, x_2, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{2-n}{2}}. \qquad (*)$$

(a) Show that $f$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad \text{for} \quad (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0). \qquad (20\%)$$

(b) Evaluate $\dfrac{\partial^2 f(x_1, x_2, \ldots, x_n)}{\partial x_3 \partial x_2}$. (5%)

(c) If $n = 2$ in $(*)$, determine the value of $f(1, 3)$. (5%)

1.(a) Exactly the same as before.

(b) $\dfrac{\partial f}{\partial x_2} = (2-n)x_2(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}}$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = (2-n)x_2(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1}\left(-\frac{n}{2}\right)\underbrace{\frac{\partial}{\partial x_3}(x_1^2 + \cdots + x_n^2)}_{2x_3}$$

$$= -(2-n)\, n\, x_2 x_3 (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1}.$$

(c) $f(x_1, x_2) = (1^2 + 3^2)^{\frac{2-2}{2}} = 10^0 = 1.$

Name________________________

2. Let $a > 0$ be a given constant. A quadric surface is given as

$$ax^2 + y^2 - az = 0.$$

(a) Find the tangent plane to the surface at the point $(a, a, a + a^2)$. (10%)

(b) Use differentials to compute an approximate value for

$$\sqrt[3]{6(9.994)^2 + 25(15.91)} \qquad (15\%)$$

with 4 decimal place accuracy.

(a) $z = \frac{1}{a}(ax^2+y^2) = x^2 + \frac{1}{a}y^2$.

$\nabla z = (2x, \frac{2}{a}y) = (2a, \frac{2}{a}\cdot a) = (2a, 2)$ when $(x,y) = (a,a)$.

So the eq'n for the tangent plane is

$$z - (a+a^2) = (2a, 2)\cdot(x-a, y-a) = 2a(x-a) + 2(y-a),$$

i.e., $2ax + 2y - z = a^2 + a$.

(b) $\begin{cases} \Delta x = 9.994 - 10 = -0.006 \\ \Delta y = 15.91 - 16 = -0.09 \end{cases}$

$f(x,y) = \sqrt[3]{6x^2+25y} = \sqrt[3]{6\cdot(10)^2 + 25\cdot 16} = \sqrt[3]{600+400} = 10$, at $(x,y) = (10,16)$.

$$f(9.994, 15.91) - f(10,16) \approx \frac{\partial f}{\partial x}(10,16)\,\Delta x + \frac{\partial f}{\partial y}(10,16)\,\Delta y$$

$\frac{\partial f}{\partial x} = \frac{1}{3}(6x^2+25y)^{-\frac{2}{3}}\cdot 12x$ at $(x,y) = (10,16) \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \frac{1}{3}\,\frac{1}{10^2}\cdot 12\cdot(10) \\ \frac{\partial f}{\partial y} = \frac{1}{3}\,\frac{1}{10^2}\cdot 25. \end{cases}$

$\frac{\partial f}{\partial y} = \frac{1}{3}(6x^2+25y)^{-\frac{2}{3}}\cdot 25$.

Hence

$$f(9.994, 15.91) \approx f(10,16) + \frac{1}{3}\cdot\frac{1}{10^2}\cdot 12(10)\cdot(-0.006) + \frac{1}{3}\cdot\frac{1}{10^2}\cdot 25\cdot(-0.09)$$

$$= 10 - 0.0024 - 0.0075 = 10 - 0.0099$$

$$= 9.9901.$$

Name________________________________

3. Given two lines

$$\left.\begin{array}{llll} L_1: & x = 2s, & y = 3 + 4s, & z = -3 + s, \\ L_2: & x = 2 - t, & y = -1 + t, & z = 5 - 3t, \end{array}\right\} \quad s, t \in \mathbb{R},$$

(a) show that $L_1$ and $L_2$ are skew lines; (7%)

(b) compute the distance between $L_1$ and $L_2$. (15%)

(a) First, note that the directions of $L_1$ and $L_2$ are, resp., $(2,4,1)$ and $(-1,1,-3)$. These two directions are not proportional. Hence $L_1$ and $L_2$ are not parallel.

Next, we determine if $L_1$ and $L_2$ intersect. Let

$$\begin{cases} 2s = 2-t \\ 3+4s = -1+t \end{cases} \qquad (\times 2): \quad \begin{array}{r} 4s = 4-2t \\ -)\ 3+4s = -1+t \\ \hline -3 = 5-3t \end{array} \Rightarrow \ 3t = 8,\ t = 8/3.$$

$$\Rightarrow s = 1 - \tfrac{t}{2} = 1 - \tfrac{4}{3} = -\tfrac{1}{3}.$$

Substituting $s = -\frac{1}{3}$, $t = 8/3$ into $z = -3+s = 5-3t$, we obtain

$-3+(-\frac{1}{3}) = 5-3(8/3) \Rightarrow -\frac{10}{3} = -3$, a contradiction. So $L_1$ and $L_2$ are skew lines.

(b) First, find a common direction $\vec{n}$ which is perpendicular to both lines:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 1 \\ -1 & 1 & -3 \end{vmatrix} = (-12-1)\,\vec{i} + (6-1)\,\vec{j} + (2+4)\,\vec{k} = -13\vec{i} + 5\vec{j} + 6\vec{k}$$

Find a plane passing $(0,3,-3)$ with normal $\vec{n}$ (so that this plane contains $L_1$):

$$-13(x-0) + 5(y-3) + 6(z+3) = 0,$$

i.e.,

$$-13x + 5y + 6z = -3.$$

Find a second plane passing $(2,-1,5)$ with normal $\vec{n}$ (so that this plane contains $L_2$ and is parallel to the first plane):

$$-13(x-2) + 5(y+1) + 6(z-5) = 0,$$

i.e.,

$$-13x + 5y + 6z = -1$$

The distance between $L_1$ and $L_2$ is the distance between the two planes. Thus

$$d = \frac{|-3-(-1)|}{\sqrt{(-13)^2+5^2+6^2}} = \frac{2}{\sqrt{230}}.$$

Name________________________________

4. (a) Let

$$f(x,y) = \frac{-7x^2y}{2x^4+3y^2}, \text{ if } (x,y) \neq (0,0).$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. (10%)

(b) Let

$$f(x,y) = \frac{-7x^4y^2}{9x^4+5y^2} + 3, \text{ if } (x,y) \neq (0,0).$$

Prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 3$.

(You must check the definition for a limit rigorously by an $(\varepsilon, \delta)$-argument.) (13%)

(a) Along the line $y=x$, we have

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{-7x^2\cdot x}{2x^4+3x^2} = \lim_{x\to 0} \frac{-7x^3}{2x^4+3x^2} = \lim_{x\to 0} \frac{-7x}{2x^2+3} = \frac{0}{3} = 0$$

Along the curve $y=x^2$, we have

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{-7x^2\cdot x^2}{2x^4+3x^4} = \lim_{x\to 0} \frac{-7}{5} = -\frac{7}{5}.$$

Since $0 \neq -\frac{7}{5}$, from the uniqueness of a limit if it exists, we conclude that the limit doesn't exist.

(b) Given any $\varepsilon > 0$, we want to show that we can choose $\delta > 0$ such that

$$|f(x,y) - 3| < \varepsilon \quad \text{if} \quad 0 < \sqrt{(x-0)^2+(y-0)^2} < \delta.$$

$$|f(x,y)-3| = \left|-\frac{7x^4y^2}{9x^4+5y^2}\right| = \left|\frac{9x^4}{9x^4+5y^2}\right| \left|\frac{7}{9}y^2\right|$$

$$\leq \left|\frac{7}{9}y^2\right| = \frac{7}{9}y^2 \leq \frac{7}{9}\left(\sqrt{x^2+y^2}\right)^2 < \frac{7}{9}\delta^2 = \varepsilon,$$

if we choose $\delta = \frac{3}{\sqrt{7}}\varepsilon$. Hence we have proved that the limit exists.