Fall 2005, Sample Test II

MATH 251 **Test II** **Name** ____________________

Section ____________

1.	______	(25%)
2.	______	(15%)
3.	______	(20%)
4.	______	(20%)
5.	______	(20%)
Total	______	(100%)

There are a total of five problems. No calculators are allowed.

1. A function $u = f(x, y)$ is said to be homogeneous of degree n for some $n > 0$ if $f(tx, ty) = t^n f(x, y)$.

   (a) Show that
   $$x f_x(x, y) + y f_y(x, y) = n f(x, y). \qquad (5\%)$$

   (b) Show that
   $$x^2 f_{xx}(x, y) + y^2 f_{yy}(x, y) + 2xy f_{xy}(x, y) = n(n-1) f(x, y). \qquad (5\%)$$

   (c) Show that $f_y(tx, ty) = t^{n-1} f_y(x, y)$ for all $t > 0$. (5%)

   (d) Show that $x f_x(2x, 2y) + y f_y(2x, 2y) = n \cdot 2^{n-1} f(x, y)$. (5%)

   (e) Let $n = 5/2$. Give a concrete example of a function $f(x, y)$ that is homogeneous of degree 5/2. (5%)

Solution:

(a) From $f(tx,ty)=t^n f(x,y)$, take $\frac{d}{dt}$:
$$\frac{d}{dt} f(tx, ty) = \frac{d}{dt}\left[t^n f(x,y)\right] = n t^{n-1} f(x,y).$$

Apply the chain rule to the left hand side above:

(*) $$\frac{d}{dt} f(tx,ty) = \frac{\partial f(tx,ty)}{\partial x}\frac{d(tx)}{dt} + \frac{\partial f(tx,ty)}{\partial y}\frac{d(ty)}{dt} = x\frac{\partial f(tx,ty)}{\partial x} + y\frac{\partial f(tx,ty)}{\partial y} = n t^{n-1} f(x,y).$$

Now set $t=1$ above. We obtain
$$x f_x(x,y) + y f_y(x,y) = n f(x,y).$$

(b) Take $\frac{d}{dt}$ of both sides of ⊛ above:

$$\frac{d^2}{dt^2} f(tx,ty) = \frac{d}{dt}\left[x\frac{\partial f(tx,ty)}{\partial x} + y\frac{\partial f(tx,ty)}{\partial y}\right] = \frac{d}{dt}\left(nt^{n-1}f(x,y)\right) = n(n-1)t^{n-2}f(x,y).$$

Apply the chain rule to the bracketed term above:

$$\frac{d}{dt}\left[x\frac{\partial f(tx,ty)}{\partial x} + y\frac{\partial f(tx,ty)}{\partial y}\right] = x\frac{d}{dt}f_x(tx,ty) + y\frac{d}{dt}f_y(tx,ty)$$

$$= x\left[f_{xx}(tx,ty)\frac{d(tx)}{dt} + f_{xy}(tx,ty)\frac{d(ty)}{dt}\right] + y\left[f_{yx}(tx,ty)\frac{d(tx)}{dt} + f_{yy}(tx,ty)\frac{d(ty)}{dt}\right]$$

$$= x\left[f_{xx}(tx,ty)x + f_{xy}(tx,ty)y\right] + y\left[f_{yx}(tx,ty)x + f_{yy}(tx,ty)y\right]$$

$$= x^2 f_{xx}(tx,ty) + 2xyf_{xy}(tx,ty) + y^2 f_{yy}(tx,ty) = n(n-1)t^{n-2}f(x,y).$$

Again, set $t=1$; we obtain

$$x^2 f_{xx}(x,y) + 2xyf_{xy}(x,y) + y^2 f_{yy}(x,y) = n(n-1)f(x,y).$$

(c)

$$\frac{\partial}{\partial y} f(tx,ty) = f_y(tx,ty)\frac{\partial(ty)}{\partial y} = tf_y(tx,ty) = \frac{\partial}{\partial y}\left[t^n f(x,y)\right] = t^n f_y(x,y).$$

In $tf_y(tx,ty) = t^n f_y(x,y)$ above, divide both sides by t. We obtain $f_y(tx,ty) = t^{n-1}f_y(x,y)$.

(d) Substitute $t=2$ in equation ⊛ on the preceding page. We obtain

$$xf_x(2x,2y) + yf_y(2x,2y) = n\cdot 2^{n-1}f(x,y).$$

(e) For example,

$$f(x,y) = 3xy^{\frac{3}{2}} - 7x^{5/2}$$

is homogeneous of degree 5.

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2. Given $f(x,y) = (x+1)^4 + 256(y-2)^4 + 16[x(y-2)+y] - 90$,

   (a) find all the critical points of f; (10%)

   (b) determine the local extrema of f. (5%)
   [Hint: Calculations will be simplified by setting $u = x+1, v = 4(y-2)$, and by writing $16[x(y-2)+y]$ in terms of u and v.]

Solution: (a) Set $u = x+1$ and $v = 4(y-2)$. Then

$$16[x(y-2)+y] = 16[x(y-2)+(y-2)+2]$$
$$= 16[(x+1)(y-2)+2] = 4\cdot[(x+1)][4(y-2)] + 32$$
$$= 4uv + 32$$

and

$$f(x,y) = u^4 + v^4 + 4uv + 32 - 90 = u^4 + v^4 + 4uv - 58 \overset{\text{def.}}{=} F(u,v).$$

We find the extremal points of $f(x,y)$ by finding the extremal points of $F(u,v)$.

$$\frac{\partial F}{\partial u} = 4u^3 + 4v = 0 \Rightarrow u^3 + v = 0$$
$$\frac{\partial F}{\partial v} = 4v^3 + 4u = 0 \Rightarrow v^3 + u = 0$$
$$\Rightarrow u = -v^3$$
$$0 = u^3 + v = (-v^3)^3 + v = -v^9 + v = 0$$

$$0 = v^9 - v = (v^4+1)(v^2+1)(v+1)(v-1)v = 0$$

Thus, $\begin{cases} v = 0, & u = 0 \\ v = 1, & u = -1 \\ v = -1, & u = 1 \end{cases}$ are 3 critical points, giving $\begin{cases} x = -1, & y = 2, \\ x = -2, & y = 9/4, \\ x = 0, & y = 7/4. \end{cases}$ respectively.

(b) $f_{xx} = 4\cdot 3\cdot (x+1)^2 \geq 0$, $f_{xy} = 16$.
$f_{yy} = 256\cdot 4\cdot 3(y-2)^2$,

(i) At $(x,y) = (-1,2)$,

$D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 16^2 = -256 < 0$.
The test is indeterminate

(ii) At $(x,y) = (-2, \frac{9}{4})$,
$D = 12\cdot(16\cdot 4\cdot 3) - 256 > 0$,
$f_{xx} = 12 > 0$
So this pt. is a local min.

(iii) at $(x,y) = (0, \frac{7}{4})$,
$D = 12\cdot(16\cdot 4\cdot 3) - 256 > 0$,
$f_{xx} = 12 > 0$.
So this pt. is a local min.

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3. (a) Find the maximum and minimum value of the function

$$f(x, y, z) = 4x + 2y + 3z$$

subject to the constraints

$$2x + 2y + z = 10, \quad 4(y+1)^2 + z^2 = 16. \qquad (14\%)$$

(b) What is the name of the surface $4(y+1)^2 + z^2 = 16$? (3%)

(c) Give a sketch of the constraint set. (3%)

(a) Let $g(x,y,z) = 2x+2y+z-10$, $h(x,y,z) = 4(y+1)^2+z^2-16$

Then $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

$$\begin{cases} 4 = \lambda\cdot 2 + \mu\cdot 0, \\ 2 = \lambda\cdot 2 + \mu\cdot 8(y+1), \\ 3 = \lambda\cdot \cancel{2}1 + \mu\cdot 2z. \end{cases}$$

$\lambda = 2$; $\begin{cases} 2 = 4 + \mu\cdot 8(y+1) \\ 3 = 4 + \mu\cdot 2z \end{cases} \Rightarrow \begin{matrix} y+1 = -\frac{1}{4}\frac{1}{\mu} \\ z = \frac{1}{2}\frac{1}{\mu} \end{matrix}$

$16 = 4(y+1)^2 + z^2 = 4\cdot\left(-\frac{1}{4}\frac{1}{\mu}\right)^2 + \left(-\frac{1}{2}\frac{1}{\mu}\right)^2 = \left(\frac{1}{4} + \frac{1}{4}\right)\frac{1}{\mu^2} = \frac{1}{2\mu^2}$

$\mu = \pm(4\sqrt{2})^{-1} = \pm\frac{\sqrt{2}}{8}$

(i) $\mu = \frac{\sqrt{2}}{8} \Rightarrow y = -1 - \frac{1}{4}\frac{8}{\sqrt{2}} = -1-\sqrt{2}$, $z = -\frac{1}{2}\frac{8}{\sqrt{2}} = +2\sqrt{2}$
$x = \frac{1}{2}(10-2y-z) = 5 - (-1-\sqrt{2}) - \frac{1}{2}(+2\sqrt{2}) = 6$ $\Big\}$ $(x,y,z) = (6, -1-\sqrt{2}, 2\sqrt{2})$

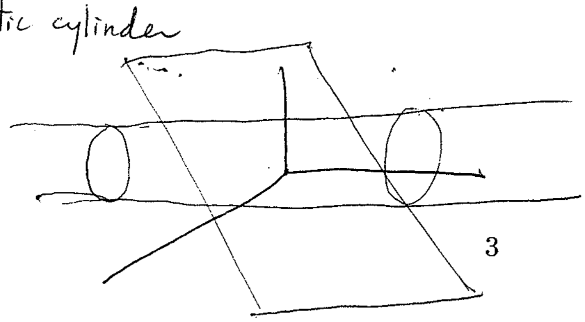
$f(x,y,z) = 4\cdot 6 + 2(-1-\sqrt{2}) + 3(2\sqrt{2}) = 22 + 4\sqrt{2}$ (max)

(ii) $\mu = -\frac{\sqrt{2}}{8} \Rightarrow y = -1 + \frac{1}{4}\frac{8}{\sqrt{2}} = -1+\sqrt{2}$, $z = \frac{1}{2}\left(-\frac{\sqrt{2}}{8}\right) = -2\sqrt{2}$
$x = \frac{1}{2}(10-2y-z) = \frac{1}{2}(10) - (-1+\sqrt{2}) - \frac{1}{2}(-2\sqrt{2}) = 6$ $\Big\}$ $(x,y,z) = (6, -1+\sqrt{2}, -2\sqrt{2})$

$f(x,y,z) = 4\cdot 6 + 2(-1+\sqrt{2}) + 3(-2\sqrt{2}) = 22 - 4\sqrt{2}$ (min)

(b) Elliptic cylinder

(c)



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4. (a) Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx$ for $a>0, b\in\mathbb{R}$ and $c>0$. (8%)

(b) Use part (a) above to determine $\displaystyle\int_{0}^{\infty} e^{-(ax^2+c)}dx$, for $a>0$. (4%)

(c) Use part (b) above to evaluate $\displaystyle\int_{0}^{\infty} x^2 e^{-2x^2}dx$. (4%)

(d) In part (a), can we evaluate the integral if $c<0$? (4%)

(a) Consider $\displaystyle\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx \cdot \int_{-\infty}^{\infty} e^{-(ay^2+by+c)}dy = \int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2+(\frac{b^2}{4a}-c)}dx \cdot \int_{-\infty}^{\infty} e^{-a(y+\frac{b}{2a})^2+(\frac{b^2}{4a}-c)}dy$

(by the change of variables $x+\frac{b}{2a}\to x$, $y+\frac{b}{2a}\to y$)

$$= \int_{-\infty}^{\infty} e^{-ax^2}dx \cdot e^{\frac{b^2}{4a}-c} \cdot \int_{-\infty}^{\infty} e^{-ay^2}dy \cdot e^{\frac{b^2}{4a}-c} = e^{2(\frac{b^2}{4a}-c)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}dx\,dy$$

(use polar coordinates $\Rightarrow$) $= e^{2(\frac{b^2}{4a}-c)} \displaystyle\int_0^{2\pi}\int_0^{\infty} e^{-ar^2} r\,dr\,d\theta$

$$= e^{2(\frac{b^2}{4a}-c)} \cdot 2\pi \cdot \left[-\frac{1}{2a}e^{-ar^2}\Big|_{r=0}^{r=\infty}\right] = e^{2(\frac{b^2}{4a}-c)} \cdot \frac{2\pi}{2a} = e^{2(\frac{b^2}{4a}-c)}\frac{\pi}{a}.$$

Thus $\displaystyle\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{e^{2(\frac{b^2}{4a}-c)}\frac{\pi}{a}} = e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}}.$

(b) By symmetry of the integrand $e^{-(ax^2+c)}$ about $x=0$, we have

$$\int_0^{\infty} e^{-(ax^2+c)}dx = \frac{1}{2}e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}}.$$

(c) $\displaystyle\int_0^{\infty} x^2 e^{-ax^2}dx = \int_0^{\infty}\left(-\frac{1}{a}\frac{\partial}{\partial a}e^{-ax^2}\right)dx = -\frac{1}{a}\frac{\partial}{\partial a}\int_0^{\infty} e^{-ax^2}dx = -\frac{1}{a}\frac{\partial}{\partial a}\left[\frac{1}{2}\sqrt{\frac{\pi}{a}}\right]$

$= \dfrac{1}{2a}\sqrt{\dfrac{\pi}{a^3}}$. Now use $a=2$, giving

$$\int_0^{\infty} x^2 e^{-2x^2}dx = \frac{1}{2(2)}\sqrt{\frac{\pi}{2^3}} = \frac{1}{8}\sqrt{\frac{\pi}{2}}.$$

(d) c is only a constant and e^{-c} is a constant. So the integral $\displaystyle\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx$ is finite regardless of the sign of c.

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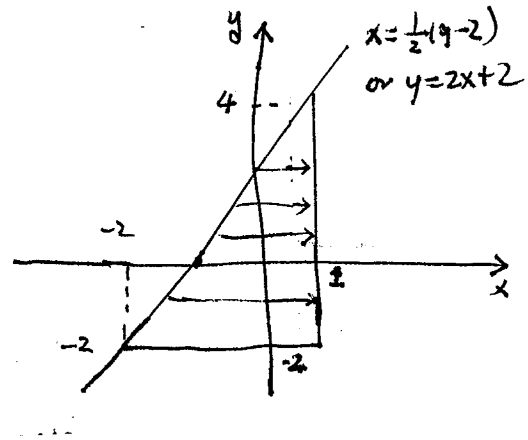
5. Consider

$$\int_{-2}^{4}\int_{\frac{1}{2}(y-2)}^{1} \cos(3x^2+12x-5)\,dx\,dy.$$

(a) Sketch the region where the integral is evaluated. (5%)

(b) Evaluate the integral by changing the order of integration. (15%)



$$\int_{-2}^{4}\int_{\frac{1}{2}(y-2)}^{1} \cos(3x^2+12x-5)\,dx\,dy$$

$$=\int_{-2}^{1}\int_{-2}^{2x+2} \cos(3x^2+12x-5)\,dy\,dx$$

$$=\int_{-2}^{1} \cos(3x^2+12x-5)\left[\,y\Big|_{y=-2}^{y=2x+2}\right]dx$$

$$=\int_{-2}^{1} \cos(3x^2+12x-5)\left[(2x+2)-(-2)\right]dx$$

$$=\int_{-2}^{1} (2x+4)\cos(3x^2+12x-5)\,dx$$

$$=\frac{1}{3}\sin(3x^2+12x-5)\Big|_{x=-2}^{x=1}$$

$$=\frac{1}{3}\left[\sin 10-\sin(-17)\right]=\frac{1}{3}\left[\sin 10+\sin 17\right].$$