MATH 251 **Fall 2005** **Test III** **November 18, 2005**

Answer Key

Name ______________________________
Section
508: (8:00-9:15)
509: (9:35-10:50)

1. ______ (15%)
2. ______ (20%)
3. ______ (20%)
4. ______ (30%)
5. ______ (15%)
Total ______ (100%)

There are a total of 5 problems.

1. If $\boldsymbol{F}(x,y,z) = [-7e^{-y\sin(z^2)}+\sin(z^2)]\boldsymbol{i}+[7x\sin(z^2)e^{-y\sin(z^2)}+4y]\boldsymbol{j}+2(7ye^{-y\sin(z^2)}+1)xz\cos(z^2)\boldsymbol{k}$, find

   (a) a function f such that $\nabla f = \boldsymbol{F}$. (12%)

   (b) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, where C is described by $\vec{r}(t) = (t, \cos(\pi t), \sin(\pi t))$, $0 \le t \le 1$. (3%)

Sol'n

(a) $\frac{\partial f}{\partial x} = -7e^{-y\sin(z^2)} + \sin(z^2) \Rightarrow f(x,y,z) = -7xe^{-y\sin(z^2)} + x\sin(z^2) + h_1(y,z)$;

$\frac{\partial f}{\partial y} = 7x\sin(z^2)e^{-y\sin(z^2)} + 4y \Rightarrow f(x,y,z) = -7xe^{-y\sin(z^2)} + 2y^2 + h_2(x,z)$;

$\frac{\partial f}{\partial z} = 2\left[7ye^{-y\sin(z^2)} + 1\right]xz\cos(z^2) \Rightarrow f(x,y,z) = -7xe^{-y\sin(z^2)} + x\sin(z^2) + h_3(x,y)$.

By comparing and combining the above three $f(x,y,z)$, we obtain

$f(x,y,z) = -7xe^{-y\sin(z^2)} + 2y^2 + x\sin(z^2) + C.$

(b) Since $\vec{F} = \nabla f$, the integral $\int \vec{F}\cdot d\vec{r}$ is path indep. So by the Fundamental Theorem of Line Integrals,

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\text{terminal pt.}) - f(\text{initial pt.})$.

initial pt., $t=0$, $\vec{r}(0) = (0, \cos(\pi\cdot 0), \sin(\pi\cdot 0)) = (0,1,0)$

terminal pt., $t=1$, $\vec{r}(1) = (0, \cos(\pi\cdot 1), \sin(\pi\cdot 1)) = (1,-1,0)$. So

$\int_C \vec{F}\cdot d\vec{r} = \left[-7(1)e^{1\cdot\sin 0} + 2(-1)^2 + 1\cdot\sin(0)\right] - \left[-7(0)e^{-1\cdot\sin 0} + 2(1)^2 + 0\cdot\sin 0\right]$

$= [-7+2] - [2] = -7.$

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2. (a) Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F}(x, y, z) = (yz - 1)\boldsymbol{i} + xz\boldsymbol{j} + x(y - 1)\boldsymbol{k}$, and C is the twisted cubic given by

$$x = t, \quad y = t^2, \quad z = t^3; \qquad 0 \le t \le 2. \qquad (15\%)$$

(b) Is the above line integral independent of path? Explain why or why not. (5%)

Sol'n (a) $\int_C \vec{F} \cdot d\vec{r} = \int_C (yz-1)dx + xz\,dy + x(y-1)dz$

$= \int_0^2 \left[(t^2 \cdot t^3 - 1)dt + t \cdot t^3 \cdot 2t\,dt + t(t^2-1)3t^2 dt\right]$

$= \int_0^2 \left[t^5 - 1 + 2t^5 + 3t^5 - 3t^3\right]dt = \int_0^2 \left[6t^5 - 3t^3 - 1\right]dt$

$= \left[t^6 - \frac{3}{4}t^4 - t\right]\Big|_0^2 = \left[2^6 - \frac{3}{4}(2)^4 - 2\right] - [0] = 64 - 12 - 2 = 50$

(b) Here, because all 3 components of $\vec{F}$ are continuously differentiable, $\int \vec{F} \cdot d\vec{r}$ is indep. of path if and only if $\nabla \times \vec{F} = \vec{0}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz-1 & xz & xy-x \end{vmatrix}$$

$= \left[\frac{\partial}{\partial y}(xy-x) - \frac{\partial}{\partial z}(xz)\right]\vec{i} - \left[\frac{\partial}{\partial x}(xy-x) - \frac{\partial}{\partial z}(yz-1)\right]\vec{j} + \left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz-1)\right]\vec{k}$

$= [x-x]\vec{i} - [y-1-y]\vec{j} + [z-z]\vec{k} = \vec{j} \neq \vec{0}.$

Therefore, the line integral is not indep. of path.

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3. Use change of variables to evaluate the integral

$$\iint_D e^{\frac{3x-y}{3x+y}}\,dA,$$

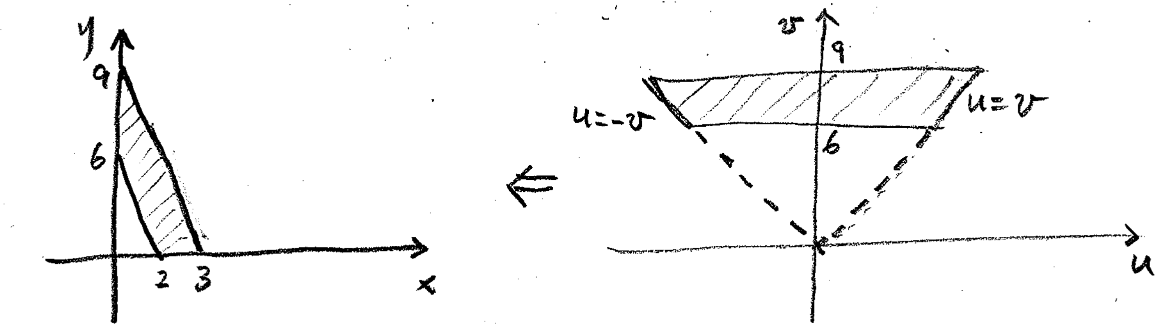
where R is the trapezoidal region with vertices (2,0), (3,0), (0,6) and (0,9). (15%)
Sketch the regions of transformation. (5%)

Sol'n

Make the change of variables

$$\begin{cases} u=3x-y\\ v=3x+y\end{cases} \Rightarrow \begin{cases} x=\frac{1}{6}(u+v)\\ y=-\frac{1}{2}(u-v)\end{cases}$$

$$\text{Jacobian}=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1/6 & 1/6\\ -1/2 & 1/2\end{vmatrix}=\frac{1}{12}-\left(-\frac{1}{12}\right)=\frac{1}{6}$$



$$\iint_D e^{\frac{3x-y}{3x+y}}\,dA=\int_6^9\int_{-v}^{v} e^{\frac{u}{v}}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$=\frac{1}{6}\int_6^9\left[\int_{-v}^{v} e^{u/v}\,du\right]dv=\frac{1}{6}\int_6^9\left[v e^{u/v}\Big|_{u=-v}^{u=v}\right]dv$$

$$=\frac{1}{6}\int_6^9 v\left(e^{v/v}-e^{-v/v}\right)dv=\frac{1}{6}\int_6^9 v(e-e^{-1})\,dv$$

$$=\frac{1}{6}(e-e^{-1})\left[\frac{1}{2}v^2\Big|_{v=6}^{v=9}\right]=\frac{1}{12}(e-e^{-1})(81-36)$$

$$=\frac{45}{12}(e-e^{-1})=\frac{15}{4}(e-e^{-1}).$$

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4. Given the circular helix

$$\left.\begin{array}{l} x(t) = 2\cos(at) \\ y(t) = 2\sin(at) \\ z(t) = bt \end{array}\right\} \quad 0 \le t \le 2\pi/a; a, b > 0,$$

(i) Let $s(t)$ be the arclength variable of the circular helix:

$$s = s(t) = \int_0^t ds.$$

Determine $s(t)$ as a function of t. (7.5%)

(ii) Determine the unit tangent vector $\vec{T}(\vec{r})$ on the curve where $\vec{r} = (-2, 0, \frac{b\pi}{a})$. (7.5%)

(iii) Evaluate the line integral $\int_C y \sin z \, ds$, where $b = a$ while $a > 0$ is arbitrary, by using the s variable. (7.5%)

(iv) Repeat part (iii) above, but by using the t variable. (7.5%).

Sol'n

(i) $s = s(t) = \int_0^t ds = \int_0^t |\vec{r}'(\tau)| d\tau$, $\vec{r}'(\tau) = (-2a\sin(a\tau), 2a\cos(a\tau), b)$

$= \int_0^t \sqrt{(-2a\sin(a\tau))^2 + (2a\cos(a\tau))^2 + b^2}\, d\tau = \int_0^t \sqrt{4a^2+b^2}\, d\tau = \sqrt{4a^2+b^2}\, t.$

(ii) $\vec{T}(\vec{r}) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$. Here $t = \frac{\pi}{a}$. So

$\vec{r}'(\frac{\pi}{a}) = (-2a\sin(a\cdot\frac{\pi}{a}), 2a\cos(a\cdot\frac{\pi}{a}), b) = (0, 2a, b),$

$\vec{T}(\vec{r}) = (0, 2a, b) / \sqrt{0^2 + (2a)^2 + b^2} = \frac{1}{\sqrt{4a^2+b^2}}(0, 2a, b).$

(iii) $\int_C y \sin z\, ds = \int 2\sin(at)\sin(at)\, ds = \int_0^L 2\sin(a\cdot\frac{s}{\sqrt{4a^2+a^2}})\sin(a\cdot\frac{s}{\sqrt{4a^2+a^2}})\, ds$

(L = total arclength $= \sqrt{4a^2+a^2}\cdot\frac{2\pi}{a}$) $\Rightarrow = 2\int_0^{\sqrt{5}a\cdot\frac{2\pi}{a}} \sin^2(a\cdot\frac{s}{\sqrt{5}a})\, ds$

$= \int_0^{2\sqrt{5}\pi} \left[1 - \cos(\frac{2as}{\sqrt{5}})\right] ds = 2\sqrt{5}\pi - \frac{\sqrt{5}}{2a}\sin(\frac{2as}{\sqrt{5}})\Big|_{s=0}^{s=2\sqrt{5}\pi}$

$= 2\sqrt{5}\pi.$

(iv) $\int_C y \sin z\, ds = \int_0^{2\pi/a} 2\sin(at)\sin at)\sqrt{5}a\, dt = \sqrt{5}a\int_0^{2\pi/a}[1 - \cos(2at)]\, dt$

$= \sqrt{5}a\left[\frac{2\pi}{a}\right] - \frac{\sqrt{5}a}{2a}\sin(2at)\Big|_{t=0}^{t=2\pi/a} = 2\sqrt{5}\pi.$

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5. Use spherical coordinates to **find the volume of the** solid that lies above the cone $z = [(x^2+y^2)/3]^{1/2}$ and below the sphere $x^2+y^2+z^2 = \begin{cases} 4z. \text{ (yellow form)} \\ 2z \text{ (white form)} \end{cases}$
(12%)
Sketch the region of integration.
(3%)

Sol'n (yellow form)

$z = \sqrt{\frac{x^2+y^2}{3}} \Rightarrow \frac{\sqrt{x^2+y^2}}{z} = \sqrt{3}$

$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}} \Rightarrow \frac{\rho\sin\phi}{\rho\cos\phi} = \sqrt{3}$

$\Rightarrow \tan\phi = \sqrt{3} \Rightarrow \phi = \frac{\pi}{3}$

$x^2+y^2+z^2 = 4z$

$\rho^2 = 4\rho\cos\phi$

$\rho = 0$ or $\rho = 4\cos\phi$

$\text{Volume} = \iiint_D dV$

$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$

$= 2\pi \int_0^{\pi/3} \left[\frac{1}{3}\rho^3 \Big|_{\rho=0}^{\rho=4\cos\phi}\right]\sin\phi \, d\phi$

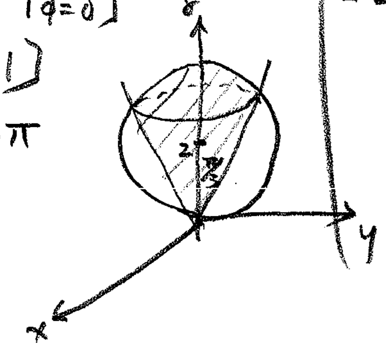
$= \frac{2\pi}{3}\int_0^{\pi/3}\left[(4\cos\phi)^3 - 0\right]\sin\phi \, d\phi$

$= \frac{2\pi}{3}\cdot 64 \cdot \int_0^{\pi/3}\cos^3\phi\sin\phi \, d\phi$

$= \left(\frac{128\pi}{3}\right)\cdot\left[-\frac{1}{4}\cos^4\phi \Big|_{\phi=0}^{\pi/3}\right]$

$= -\frac{32\pi}{3}\left[\cos^4\left(\frac{\pi}{3}\right) - 1\right]$

$= -\frac{32\pi}{3}\cdot\left[\frac{1}{16} - 1\right] = 10\pi$



Sol'n (white form)

same as left

$x^2+y^2+z^2 = 2z$

$\rho^2 = 2\rho\cos\phi$

$\rho = 0$ or $\rho = 2\cos\phi$

$\text{volume} = \iiint_D dV$

$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{2\cos\phi} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$

$= 2\pi\int_0^{\pi/3}\left[\frac{1}{3}\rho^3 \Big|_{\rho=0}^{\rho=2\cos\phi}\right]\sin\phi \, d\phi$

$= \frac{2\pi}{3}\int_0^{\pi/3}\left[8\cos^3\phi\right]\sin\phi \, d\phi$

$= \frac{16\pi}{3}\left[-\frac{1}{4}\cos^4\phi \Big|_0^{\pi/3}\right] = -\frac{4\pi}{3}\left[-\frac{15}{16}\right]$

$= \frac{5}{4}\pi.$