MATH 251 Test II October 21, 2005 Name ______________

Section ____________
508: 8-9:15 am
509: 9:35-10:50 am

1.	______	(25%)
2.	______	(15%)
3.	______	(20%)
4.	______	(20%)
5.	______	(20%)
Total	______	(100%)

There are a total of five problems. No calculators are allowed.

1. (a) Find the maximum and minimum value of the function

$$f(x, y, z) = 3z - 2x - 4y$$

subject to the constraints

$$z - 2x - 2y = 10, \quad 4(1-x)^2 + z^2 = 16. \qquad (14\%)$$

(b) What is the name of the surface $4(1-x)^2 + z^2 = 16$? (3%)

(c) Give a sketch of the constraint set. (3%)

Solution (a) $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$, here $g_1 = z - 2x - 2y - 10 = 0$, $g_2 = 4(x-1)^2 + z^2 - 16 = 0$.

$$\begin{cases} -2 = \lambda(-2) + \mu(8)\cdot(x-1) \\ -4 = \lambda\cdot(-2) + \mu\cdot 0 \\ 3 = \lambda\cdot 1 + \mu\cdot(2z) \end{cases} \Rightarrow \lambda = 2 \Rightarrow \begin{cases} 8\mu(x-1) = 2 \\ 2\mu z = 1 \end{cases} \Rightarrow \begin{cases} x - 1 = \frac{1}{4\mu} \\ z = \frac{1}{2\mu} \end{cases}$$

Hence $4(x-1)^2 + z^2 = 4\cdot\frac{1}{16\mu^2} + \frac{1}{4\mu^2} = \frac{1}{2\mu^2} = 16 \Rightarrow \mu^2 = \frac{1}{32} \Rightarrow \mu = \pm\frac{\sqrt{2}}{8}$.

When

(1) $\mu = \frac{\sqrt{2}}{8}$, $x = 1 + \frac{1}{\sqrt{2}/2} = 1 + \sqrt{2}$, $z = \frac{1}{2\cdot(\sqrt{2}/8)} = 2\sqrt{2}$.

So $y = \frac{1}{2}(z - 2x - 10) = \frac{1}{2}[2\sqrt{2} - 2(1+\sqrt{2}) - 10] = -6$

$f(1+\sqrt{2}, -6, 2\sqrt{2}) = 3(2\sqrt{2}) - 2(1+\sqrt{2}) - 4(-6) = 22 + 4\sqrt{2}$ (max)

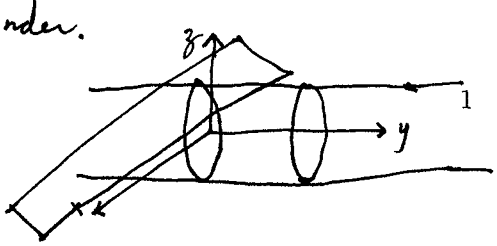
(2) $\mu = -\frac{\sqrt{2}}{8}$, $x = 1 - \frac{1}{\sqrt{2}/2} = 1 - \sqrt{2}$, $z = -\frac{1}{2\cdot(\sqrt{2}/8)} = -2\sqrt{2}$.

So $y = \frac{1}{2}(z - 2x - 10) = \frac{1}{2}[-2\sqrt{2} - 2(1-\sqrt{2}) - 10] = -6$,

$f(1-\sqrt{2}, -6, -2\sqrt{2}) = 3(-2\sqrt{2}) - 2(1-\sqrt{2}) - 4(-6) = 22 - 4\sqrt{2}$ (min).

(b) Elliptic cylinder.

(c)



Name ____________________________

2. Given $f(x,y) = \dfrac{1}{256}(x-1)^4 + (y+2)^4 + xy + 2x - y + 1$,

(a) find all the critical points of f; (10%)

(b) determine the local extrema of f. (5%)

Solution $f(x,y) = \left[\frac{1}{4}(x-1)\right]^4 + (y+2)^4 + 4\cdot\left[\frac{1}{4}(x-1)\right](y+2) + 3.$

Let $u = \frac{1}{4}(x-1)$, $v = y+2$. Then $f = u^4 + v^4 + 4uv + 3$.

To find the critical points, set

$$\begin{cases} \dfrac{\partial f}{\partial u} = 4u^3 + 4v = 0 \\ \dfrac{\partial f}{\partial v} = 4v^3 + 4u = 0 \end{cases} \Rightarrow \begin{cases} v = -u^3 \\ v^3 + u = 0 \end{cases} \Rightarrow (-u^3)^3 + u = 0$$

$$u^9 - u = 0$$

$$u(u^4+1)(u^2+1)(u-1)(u+1) = 0$$

So $u = 0$, or $u = 1$, or $u = -1$.

(1) When $u = 0$, $v = 0$. So $\begin{cases} \frac{1}{4}(x-1) = 0 \\ y+2 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -2 \end{cases}$

(2) When $u = 1$, $v = -1$. So $\begin{cases} \frac{1}{4}(x-1) = 1 \\ y+2 = -1 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = -3 \end{cases}$

(3) When $u = -1$, $v = 1$. So $\begin{cases} \frac{1}{4}(x-1) = -1 \\ y+2 = 1 \end{cases} \Rightarrow \begin{cases} x = -3 \\ y = -1 \end{cases}$

There are 3 sets of critical points: $(1,-2)$, $(5,-3)$, $(-3,-1)$.

(b) The local extrema are determined by the discriminant

$$D = f_{xx}f_{yy} - f_{xy}^2.$$

$$f_{xx} = \frac{12}{256}(x-1)^2 = \frac{3}{64}(x-1)^2,\quad f_{yy} = 12(y+2)^2,\quad f_{xy} = 1.$$

(i) For $(1,-2)$,

$$D = 0\cdot 0 - 1^2 = -1 < 0,\quad f_{xx} = 0.$$

So it is indeterminate.

(ii) For $(5,-3)$,

$$D = \left(\frac{3}{64}\cdot 16\right)(12\cdot 1^2) - 1^2 = 9 - 1 = 8 > 0,\ f_{xx} > 0.$$

So $(5,-3)$ is a local min.

(iii) For $(-3,-1)$,

$$D = \left(\frac{3}{64}\cdot 16\right)(12\cdot 1^2) - 1^2 = 9 - 1 = 8 > 0,\ f_{xx} > 0.$$

So $(-3,-1)$ is a local min.

Name ____________________________

3. A function $u = f(x,y)$ is said to be homogeneous of degree n for some n if $f(tx,ty) = t^n f(x,y)$.

   (a) Show that
   $$xf_x(x,y) + yf_y(x,y) = nf(x,y). \qquad (5\%)$$

   (b) Show that
   $$x^2 f_{xx}(x,y) + y^2 f_{yy}(x,y) + 2xyf_{xy}(x,y) = n(n-1)f(x,y). \qquad (5\%)$$

   (c) Show that $f_y(tx,ty) = t^{n-1}f_y(x,y)$ for all $t > 0$. (5%)

   (d) Let $n = -3/2$. Give a concrete example of a function $f(x,y)$ that is homogeneous of degree $-3/2$. (5%)

   (e) Let $n = 1$. Show that $x^2 f_{xx}(x,y) + 2xyf_{xy}(x,y) + y^2 f_{yy}(x,y) = 0$.

Solution (a) $\frac{d}{dt} f(tx,ty) = \frac{d}{dt}\left[t^n f(x,y)\right] = nt^{n-1}f(x,y)$

Apply the chain rule to the left side:

$$(*) \quad \frac{d}{dt} f(tx,ty) = f_x(tx,ty)\frac{d}{dt}(tx) + f_y(tx,ty)\frac{d}{dt}(ty) = f_x(tx,ty)\,x + f_y(tx,ty)\,y = nt^{n-1}f(x,y).$$

Setting $t=1$, we obtain
$$xf_x(x,y) + y f_y(x,y) = nf(x,y).$$

(b) $$\frac{d^2}{dt^2} f(tx,ty) = \frac{d^2}{dt^2}\left[t^n f(x,y)\right] = n(n-1)f(x,y).$$

From $(*)$ above, we have

$$\frac{d^2}{dt^2} f(tx,ty) = \frac{d}{dt}\left[f_x(tx,ty)x + f_y(tx,ty)y\right] = \left[f_{xx}(tx,ty)\frac{d}{dt}(tx) + f_{xy}(tx,ty)\frac{d}{dt}(ty)\right]x$$
$$+ \left[f_{yx}(tx,ty)\frac{d}{dt}(tx) + f_{yy}(tx,ty)\frac{d}{dt}(ty)\right]y = x^2 f_{xx}(tx,ty) + 2xyf_{xy}(tx,ty) + y^2 f_{yy}(tx,ty).$$

Setting $t=1$ again, we obtain

$$(**) \quad x^2 f_{xx}(x,y) + 2xyf_{xy}(x,y) + y^2 f_{yy}(x,y) = n(n-1)f(x,y).$$

(c) $\frac{\partial}{\partial y}\left[f(tx,ty)\right] = \frac{\partial}{\partial y}\left[t^n f(x,y)\right] = t^n f_y(x,y)$

$$f_y(tx,ty)\frac{\partial}{\partial y}(ty) = t\,f_y(tx,ty) = t^n f_y(x,y).$$

Divide both sides by t:
$$f_y(tx,ty) = t^{n-1} f_y(x,y).$$

(d) $f(x,y) = x^{-3/2} - 3xy^{-5/2}$, for example.

(e) From $(**)$ above, we have

$$x^2 f_{xx}(x,y) + 2xyf_{xy}(x,y) + y^2 f(x,y) = 1\cdot(1-1)f(x,y) = 0.$$

Name ______________________

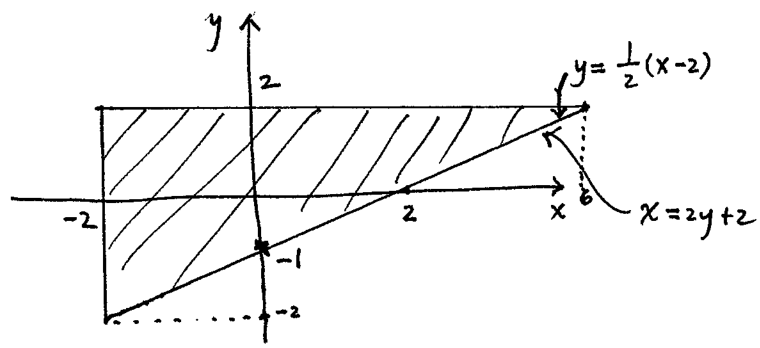
4. Consider

$$\int_{-2}^{2}\int_{-2}^{2y+2} \sin(-x^2+12x+5)\,dx\,dy.$$

(a) Sketch the region where the integral is evaluated. (5%)

(b) Evaluate the integral by changing the order of integration. (15%)

Solution (a)



(b)

$$\int_{-2}^{2}\int_{-2}^{2y+2} \sin(-x^2+12x+5)\,dx\,dy = \int_{-2}^{6}\int_{\frac{1}{2}(x-2)}^{2} \sin(-x^2+12x+5)\,dy\,dx$$

$$= \int_{-2}^{6}\left[\,y\Big|_{y=\frac{1}{2}(x-2)}^{2}\right]\sin(-x^2+12x+5)\,dx$$

$$= \int_{-2}^{6}\left[2-\left(\tfrac{1}{2}x-2\right)\right]\sin(-x^2+12x+5)\,dx$$

$$= \int_{-2}^{6}\left[3-\tfrac{1}{2}x\right]\sin(-x^2+12x+5)\,dx$$

$$= \frac{1}{4}\int_{-2}^{6}\left[-2x+12\right]\sin(-x^2+12x+5)\,dx$$

$$= -\frac{1}{4}\left[\cos(-x^2+12x+5)\Big|_{x=-2}^{6}\right]$$

$$= -\frac{1}{4}\left[\cos(-36+72+5)-\cos(-4-24+5)\right]$$

$$= -\frac{1}{4}\left[\cos(41)-\cos(-23)\right] = \frac{1}{4}\left(\cos 23-\cos 41\right).$$

Name ______________________

5. (a) Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx$ for $a>0, b\in\mathbb{R}$ and $c>0$. (8%)

(b) Use part (a) above to determine $\displaystyle\int_{0}^{\infty} e^{-(ax^2+c)}dx$, for $a>0$. (4%)

(c) Use part (b) above to evaluate $\displaystyle\int_{-\infty}^{0} x^4 e^{-2x^2}dx$. (4%)

(d) In part (a), can we evaluate the integral if $a<0$? (Answer yes or no and explain. No partial credit if your answer is wrong.) (4%)

Solution (a) Consider

$$\left[\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx\right]^2 = \int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx\cdot\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx$$

$$= \int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx\cdot\int_{-\infty}^{\infty} e^{-(ay^2+by+c)}dy = \int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2+(\frac{b^2}{4a}-c)}dx\cdot\int_{-\infty}^{\infty} e^{-a(y+\frac{b}{2a})^2+(\frac{b^2}{4a}-c)}dy$$

$$=\left[e^{\frac{b^2}{4a}-c}\cdot e^{\frac{b^2}{4a}-c}\right]\int_{-\infty}^{\infty} e^{-a(x+\frac{b}{2a})^2}dx\cdot\int_{-\infty}^{\infty} e^{-a(y+\frac{b}{2a})^2}dy = e^{2(\frac{b^2}{4a}-c)}\int_{-\infty}^{\infty} e^{-ax^2}dx\cdot\int_{-\infty}^{\infty} e^{-ay^2}dy$$

$$= e^{2(\frac{b^2}{4a}-c)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}dx\,dy = e^{2(\frac{b^2}{4a}-c)}\int_0^{2\pi}\int_0^{\infty} e^{-ar^2}r\,dr\,d\theta$$

$$= e^{2(\frac{b^2}{4a}-c)}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\infty} e^{-ar^2}r\,dr\right) = e^{2(\frac{b^2}{4a}-c)}\cdot 2\pi\cdot\left(-\frac{1}{2a}e^{-ar^2}\Big|_{r=0}^{r=\infty}\right)$$

$$= e^{2(\frac{b^2}{4a}-c)}\cdot 2\pi\cdot\left(-\frac{1}{2a}\right)\Big[\underset{0}{\underset{\parallel}{e^{-\infty}}}-\underset{1}{\underset{\parallel}{e^{0}}}\Big] = \frac{\pi}{a}e^{2(\frac{b^2}{4a}-c)}.$$

Hence $\displaystyle\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}e^{2(\frac{b^2}{4a}-c)}} = e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}}$.

(b) $\displaystyle\int_0^{\infty} e^{-(ax^2+c)}dx = e^{-c}\int_0^{\infty} e^{-ax^2}dx = e^{-c}\cdot\int_{-\infty}^{\infty} e^{-ax^2}dx\cdot\frac{1}{2} = \frac{1}{2}e^{-c}\sqrt{\frac{\pi}{a}}$.

(c) $\displaystyle\int_{-\infty}^{0} e^{-ax^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

$$\frac{\partial}{\partial a}\int_{-\infty}^{0} e^{-ax^2}dx = \int_{-\infty}^{0}\left[\frac{\partial}{\partial a}e^{-ax^2}\right]dx = \int_{-\infty}^{0}(-x^2)e^{-ax^2}dx = -\int_{-\infty}^{0}x^2e^{-ax^2}dx = \frac{\partial}{\partial a}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right)$$

$$\frac{\partial^2}{\partial a^2}\int_{-\infty}^{0} e^{-ax^2}dx = \int_{-\infty}^{0}(-x^2)(-x^2)e^{-ax^2}dx = \frac{\partial^2}{\partial a^2}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\sqrt{\pi}\,a^{-5/2}$$

$\displaystyle\int_{-\infty}^{0} x^4e^{-ax^2}dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$. Set $a=2$. We obtain

$\displaystyle\int_{-\infty}^{0} x^4e^{-2x^2}dx = \frac{3}{32}\sqrt{\frac{\pi}{2}}$.

(d) No, because e^{-ax^2} will go to $+\infty$ as $x\to\pm\infty$, causing the integral to diverge.