MATH 251 Test I October 1, 2004

There are a total of 4 problems. No calculators are allowed.

Name ______________________________
Section 502 (MWF 10:20–11:10)
507 (TR 11:10–12:25)

1. ______
2. ______
3. ______
4. ______
Total ______

1. Given two lines

$$\left.\begin{array}{llll} L_1: & x=-1-3t, & y=2-t, & z=5+t, \\ L_2: & x=3+s, & y=2s, & z=-3+4s, \end{array}\right\} \quad s,t\in\mathbb{R},$$

(a) show that L_1 and L_2 are skew lines; (10%)

(b) compute the distance between them. (15%)

(a) First, note that the directions of L_1 and L_2 are, resp., $(-3,-1,1)$ and $(1,2,4)$. These two directions are not proportional. Hence L_1 and L_2 are not parallel.
Next, we determine if L_1 and L_2 intersect. Let

$$\begin{cases} -1-3t=3+s \\ 2-t=2s \end{cases} \qquad (\times 2): \quad \begin{array}{l} -2-6t=6+2s \\ \underline{-)\ \ 2-t=2s} \\ -4-5t=6 \Rightarrow -5t=10 \Rightarrow t=-2 \end{array}$$

$$\Rightarrow s=-1-3t-3=-1+6-3=2$$

Substituting $s=2,\ t=-2$ into $z=5+t=-3+4s$, we obtain

$$5+(-2)=-3+4(2) \Rightarrow 3=5,$$ a contradiction. So L_1 and L_2 are skew lines.

(b) First, find a common direction $\vec{n}$ which is perpendicular to both lines:

$$\vec{n}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = (-4-2)\vec{i}+(12+1)\vec{j}+(-6+1)\vec{k}=-6\vec{i}+13\vec{j}-5\vec{k}$$

Find a plane passing $(-1,2,5)$ with normal $\vec{n}$ (so that this plane contains L_1):

$$-6(x+1)+13(y-2)-5(z-5)=0,$$

i.e.,

$$-6x+13y-5z=6+26-25=7$$

Find a second plane passing $(3,0,-3)$ with normal $\vec{n}$ (so that this plane contains L_2 and is parallel to the first plane):

$$-6(x-3)+13(y-0)-5(z+3)=0,$$

i.e.,

$$-6x+13y-5z=-18+15=-3.$$

The distance between L_1 and L_2 is the distance between the two planes. Thus

$$d=\frac{|-3-7|}{\sqrt{(-6)^2+13^2+(-5)^2}}=\frac{10}{\sqrt{230}}$$

④

2. Let n be a positive integer greater than 2, and

$$f(x_1, x_2, \ldots, x_n) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{2-n}{2}}.$$

(a) Show that f satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0, \quad \text{for} \quad (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0). \qquad (20\%)$$

(b) Evaluate $\dfrac{\partial^2 f(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2}$. (5%)

(a) For $i = 1, 2, \cdots, n$

$$\frac{\partial f}{\partial x_i} = \left(\frac{2-n}{2}\right)(x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{2-n}{2}-1} \cdot \frac{\partial}{\partial x_i}(x_1^2 + \cdots + x_n^2)$$

$$= \left(\frac{2-n}{2}\right)(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} \cdot 2x_i$$

$$= (2-n)\, x_i\, (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} \qquad (*)$$

$$\frac{\partial^2 f}{\partial x_i^2} = (2-n)\left[(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} + \left(-\frac{n}{2}\right) x_i\, (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1} \cdot 2x_i\right]$$

$$= (2-n)\left[(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} - n x_i^2\, (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1}\right].$$

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$$= (2-n) \sum_{i=1}^{n} \left[(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} - n x_i^2 (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1}\right]$$

$$= (2-n)\left\{\sum_{i=1}^{n} (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} - n\left(\sum_{i=1}^{n} x_i^2\right)(x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1}\right\}$$

$$= (2-n)\left\{n \cdot (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}} - n \cdot (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1+1}\right\}$$

$$= 0.$$

(b) According to (*) above,

$$\frac{\partial f}{\partial x_1} = (2-n)\, x_1 (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}}.$$

Thus,

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = (2-n)\, x_1\, (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1} \cdot \left(-\frac{n}{2}\right) \underbrace{\frac{\partial}{\partial x_2}(x_1^2 + x_2^2 + \cdots + x_n^2)}_{2x_2}$$

$$= -(2-n)\, n\, x_1 x_2\, (x_1^2 + \cdots + x_n^2)^{-\frac{n}{2}-1}.$$

3. (a) Let

$$f(x,y) = \frac{-5xy^2}{x^2+3y^4}, \quad \text{if} \quad (x,y) \neq (0,0).$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. (12%)

(b) Let

$$f(x,y) = \frac{-7xy^2}{x^2+2y^2}, \quad \text{if} \quad (x,y) \neq (0,0).$$

Prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

(You must check the definition for a limit rigorously by an (ε, δ)-argument.) (13%)

3. (a)

Along the line $y=x$, we have

$$\lim_{x\to0} f(x,x) = \lim_{x\to0} \frac{-5x(x)^2}{x^2+3x^4} = \lim_{x\to0} \frac{-5x^2\cdot x}{x^2(1+3x^2)}$$

$$= \lim_{x\to0} \frac{-5x}{1+3x^2} = \frac{-5\cdot0}{1+3\cdot0} = 0.$$

Along the curve $x=y^2$, we have

$$\lim_{y\to0} f(y^2,y) = \lim_{y\to0} \frac{-5y^2\cdot y^2}{(y^2)^2+3y^4} = \lim_{y\to0} \frac{-5y^4}{y^4(1+3)} = \lim_{y\to0} \frac{-5}{4} = -\frac{5}{4}.$$

Since $0 \neq -\frac{5}{4}$, from the uniqueness of a limit if it exists, we see that the limit doesn't exist.

(b) Given any $\varepsilon>0$, we want to show that we can choose $\delta>0$ such that

$$|f(x,y)-0| < \varepsilon \quad \text{if} \quad 0 < \sqrt{(x-0)^2+(y-0)^2} < \delta.$$

$$|f(x,y)-0| = \left|\frac{-7xy^2}{x^2+2y^2}\right| = \left|\frac{2y^2}{x^2+2y^2}\right| \left|-\frac{7}{2}x\right| \leq \left|-\frac{7}{2}x\right|$$

$$= \frac{7}{2}|x| \leq \frac{7}{2}\sqrt{x^2+y^2} < \frac{7}{2}\delta = \varepsilon,$$

provided that we choose $\delta = \frac{2\varepsilon}{7}$. Hence we have proved that the limit exists.

4. (a) Let $a > 0$ be a given constant. Find the tangent plane to the surface

$$z = ax^2 + y^2 \quad \text{at the point} \quad (1, 1, a+1). \qquad (8\%)$$

What is the name of the surface given in part (a)? (2%)

(b) Use differentials to compute an approximation value for

$$\sqrt[3]{6(10.2)^2 + 400(0.98)^2},$$

with 4 decimal place accuracy. (15%)

(a) $\left.\frac{\partial z}{\partial x}\right|_{x=1,y=1} = \left.2ax\right|_{\substack{x=1\\y=1}} = 2a$

$\left.\frac{\partial z}{\partial y}\right|_{x=1,y=1} = \left.2y\right|_{\substack{x=1\\y=1}} = 2$

Thus the eq. of the tangent line is

$$z - (a+1) = 2a(x-1) + 2(y-1),$$

or $$2ax + 2y - z = a+1.$$

The surface is an elliptic paraboloid.

(b) $f(x,y) = \sqrt[3]{6x^2+400y^2}$; $\frac{\partial f}{\partial x} = \frac{1}{3}(6x^2+400y^2)^{-\frac{2}{3}} \cdot 12x$, $\frac{\partial f}{\partial y} = \frac{1}{3}(6x^2+400y^2)^{-\frac{2}{3}} \cdot 800y$

$x_0 = 10$, $y_0 = 1$

$f(10,1) = \sqrt[3]{600+400} = \sqrt[3]{1000} = 10.$

$\Delta x = 10.2 - 10 = 0.2$, $\Delta y = 0.98 - 1 = -0.02$

$$f(10.2, 0.98) \approx f(10,1) + \frac{\partial f(10,1)}{\partial x}\Delta x + \frac{\partial f(10,1)}{\partial y}\Delta y$$

$$= 10 + \frac{1}{3}(1000)^{-\frac{2}{3}} \cdot 12 \cdot 10(0.2) + \frac{1}{3}(1000)^{-\frac{2}{3}} \cdot 800 \cdot y(-0.02)$$

$$= 10 + \frac{24}{300} - \frac{16}{300}$$

$$\approx 10.0267.$$