Answer Key

MATH 601 Section 603 Test II Fall 2004 Name ______________________

There are a total of five problems.
No calculators are allowed.

1. ______ (20%)
2. ______ (20%)
3. ______ (20%)
4. ______ (20%)
5. ______ (20%)
Total score ______ (100%)

1. Let A be an $m \times n$ matrix.

   (i) Define the left and right inverses of A. (5%)

   (ii) Show that if the column vectors of A span $\mathbb{R}^m$, then A has a right inverse. (5%)

   (iii) Show that if the column vectors of A are linearly independent, then A has a left inverse. (5%)

   (iv) Show that $m = n$ if A satisfies the assumptions in both parts (i) and (ii) above? (5%)

Solution (i) A matrix L of dimension $n \times m$ satisfying $LA = I_n$ is called a left inverse of A. A matrix R of dimension $n \times m$ satisfying $AR = I_m$ is called a right inverse of A.

(ii) Let $A = [\vec{a}_1\, \vec{a}_2 \cdots \vec{a}_n]$, where $\vec{a}_j$, $1 \le j \le n$, are the column vectors. Then because $\vec{a}_1, \cdots, \vec{a}_n$ span $\mathbb{R}^m$, the standard basis vectors $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_m$ can be expressed as linear combinations of $\vec{a}_1, \cdots, \vec{a}_n$ as follows:

$$\vec{e}_j = \vec{a}_1 k_{1j} + \vec{a}_2 k_{2j} + \cdots + \vec{a}_n k_{nj}, \quad j = 1, 2, \cdots, m.$$

That is,

$$[\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1m} \\ k_{21} & k_{22} & \cdots & k_{2m} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nm} \end{bmatrix} \equiv AK = [\vec{e}_1\, \vec{e}_2 \cdots \vec{e}_m] = I_m.$$

Thus K is a right inverse of A.

(iii) If the column vectors of A are linearly indep., then rank $A = n$. Since "row rank = column rank", the number of linearly indep. rows is also equal to n.

Now consider A^T. The column vectors of A^T, which are row vectors of A, have rank n in $\mathbb{R}^n$. Thus these column vectors span $\mathbb{R}^n$. We can apply part (ii) to show that A^T has a right inverse:

$$A^T G = I_n, \quad \text{for some } m \times n \text{ matrix } G.$$

So $(A^T G)^T = I_n^T = I_n$, $G^T A = I_n$! Hence G^T is a left inverse of A.

(iv) If the column vectors of A span $\mathbb{R}^m$ and are also linearly indep., they must constitute a basis for $\mathbb{R}^m$. But a basis for $\mathbb{R}^m$ contains exactly m vectors. Hence $m=n$.

2. Let $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$L(\vec{x}) = (x_1 + 2x_2 - x_3, x_1 - 2x_2 + x_3)^T, \text{ for } \vec{x} = (x_1, x_2, x_3)^T.$$

(i) Find the matrix representation of L with respect to the ordered bases $[\vec{v}_1, \vec{v}_2, \vec{v}_3]$ and $[\vec{w}_1, \vec{w}_2]$, where

$$\vec{v}_1 = (1,0,0)^T, \vec{v}_2 = (1,1,0)^T, \vec{v}_3 = (1,1,1)^T$$

and

$$\vec{w}_1 = (1,2)^T, \vec{w}_2 = (3,1)^T \qquad (15\%)$$

(ii) State the theorem based on which you have used to solve the problem in part (i). (5%)

Solution: (i) Write the matrix

$$\left[\vec{w}_1\ \vec{w}_2 \,\vdots\, L(\vec{v}_1)\ L(\vec{v}_2)\ L(\vec{v}_3)\right] = \left[\begin{array}{cc:ccc} 1 & 3 & 1 & 3 & 2 \\ 2 & 1 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 2R_1 \end{array}$$

Perform Gauss-Jordan as shown above ($\Rightarrow$)

$$\left[\begin{array}{cc:ccc} 1 & 3 & 1 & 3 & 2 \\ 0 & -5 & -1 & -7 & -4 \end{array}\right] R_2 \div (-5) \Rightarrow \left[\begin{array}{cc:ccc} 1 & 3 & 1 & 3 & 2 \\ 0 & 1 & 1/5 & 7/5 & 4/5 \end{array}\right] \overset{R_1 - 3R_2}{\Rightarrow} \left[\begin{array}{cc:ccc} 1 & 0 & 2/5 & -6/5 & -2/5 \\ 0 & 1 & 1/5 & 7/5 & 4/5 \end{array}\right]$$

$$\underbrace{\begin{matrix} 2/5 & -6/5 & -2/5 \\ 1/5 & 7/5 & 4/5 \end{matrix}}_{A}$$

The matrix A shown above is the representation.

(ii) Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and $\{\vec{w}_1, \cdots, \vec{w}_m\}$ be a basis for $\mathbb{R}^m$. Then the matrix representation A for L with respect to the above ordered bases is obtainable from

$$\left[\vec{w}_1 \cdots \vec{w}_m \,\vdots\, L(\vec{v}_1)\ L(\vec{v}_2) \cdots L(\vec{v}_n)\right] \xrightarrow{\text{row echelon form}} \left[I_m \,\vdots\, A\right].$$

3. Let S be the subspace of $C[0,1]$ spanned by $1, e^x$ and xe^x. Define the linear operator L on S by

$$L(f(x)) = f'(x) - f(1).$$

Find the matrix representing L with respect to the ordered basis $[1, e^x, xe^x]$. (20%)

Solution

$$L(1) = 1' - 1 = -1$$

$$L(e^x) = (e^x)' - e^1 = -e^1 + e^x = -e + e^x$$

$$L(xe^x) = (xe^x)' - 1 \cdot e^1 = -e + [e^x + xe^x] = -e + e^x + xe^x.$$

Thus the matrix representation of L is

$$\begin{bmatrix} -1 & -e & -e \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Let A be an $m \times n$ matrix. Show that

(i) If $\vec{x} \in N(A^TA)$, then $Ax \in R(A) \cap N(A^T)$. (5%)

(ii) $N(A^TA) = N(A)$. (5%)

(iii) A and A^TA have the same rank. (5%)

(iv) If A has linearly independent columns, then A^TA is nonsingular. (5%)

Solution (i) If $\vec{x} \in N(A^TA)$, then $(A^TA)\vec{x} = \vec{0}$; so $A^T(A\vec{x}) = \vec{0}$, i.e., $A\vec{x} \in N(A^T)$. Also, $A\vec{x} \in R(A)$ by the very definition of $R(A)$. Hence $A\vec{x} \in N(A^T) \cap R(A)$.

(ii) We first show $N(A) \subseteq N(A^TA)$. If $\vec{x} \in N(A)$, then $A\vec{x} = \vec{0}$. So $A^T(A\vec{x}) = \vec{0}$. This gives $(A^TA)\vec{x} = \vec{0}$, i.e., $\vec{x} \in N(A^TA)$.

Next, we show $N(A^TA) \subseteq N(A)$. If $\vec{x} \in N(A^TA)$, then $A^TA\vec{x} = \vec{0}$. Therefore $0 = A^TA\vec{x} \cdot \vec{x} = A\vec{x} \cdot A\vec{x} = |A\vec{x}|^2$. This gives $A\vec{x} = \vec{0}$. Hence $\vec{x} \in N(A)$,

(iii) Since

$$\text{rank } A + \text{nullity } A = n,$$

$$\text{rank } A^TA + \text{nullity } A^TA = n,$$

by part (ii) we have nullity $A =$ nullity A^TA. Hence rank $A =$ rank A^TA.

(iv) A^TA is an $n \times n$ (square) matrix and, by assumption, rank $A = n$. Use part (iii); we see that nullity $A^TA =$ nullity $A = 0$. Therefore A^TA is an invertible matrix because A^TA doesn't have any nontrivial solution.

5. Let A be an $m \times n$ matrix of rank n. Show that $\hat{x} = (A^T A)^{-1} A^T \boldsymbol{b}$ is the unique least squares solution to $Ax = \boldsymbol{b}$. (20%)

Solution Since $\mathbb{R}^m = R(A) \oplus [R(A)]^\perp$, we write

$$\vec{b} = \vec{b}_1 + \vec{b}_2$$

where $\vec{b}_1 \in R(A)$ and $\vec{b}_2 \in R(A)^\perp$.

$\hat{x}$ is a solution to the least square problem

$$\min_{\vec{x} \in \mathbb{R}^m} |A\vec{x} - \vec{b}|^2 = |A\hat{\vec{x}} - \vec{b}|^2$$

iff $A\hat{\vec{x}} = \vec{b}_1$

because otherwise $A\hat{\vec{x}} - \vec{b}_1 \neq \vec{0}$ and

$$|A\hat{x} - \vec{b}|^2 = |A\hat{\vec{x}} - (\vec{b}_1 + \vec{b}_2)|^2 = |A\hat{\vec{x}} - \vec{b}_1|^2 + |\vec{b}_2|^2$$

$$> |\vec{b}_2|^2 = \min_{\vec{x} \in \mathbb{R}^m} |A\vec{x} - \vec{b}|,$$

which violates the least square property of $\hat{x}$.

But $A\hat{\vec{x}} = \vec{b}_1 = \vec{b} - \vec{b}_2$ is equivalent to

$$A\hat{\vec{x}} - \vec{b} = -\vec{b}_2 \in R(A)^\perp = N(A^T).$$

Thus

$$A^T(A\hat{\vec{x}} - \vec{b}) = \vec{0}$$

$$A^T A \hat{\vec{x}} = A^T \vec{b}, \quad \hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}.$$

Note that $A^T A$ is invertible by problem 4 (iv) of this test.

Also, $\hat{\vec{x}}$ is unique because $\hat{\vec{x}}$ is given explicitly as

$$\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}.$$