Sketch of solutions

**MATH 601 Section 603 Test III Fall 2004 (11/16/2004)**

**Name** ____________________

| | | |
|---|---|---|
| 1. | ______ | (20%) |
| 2. | ______ | (20%) |
| 3. | ______ | (20%) |
| 4. | ______ | (20%) |
| 5. | ______ | (20%) |
| Total score | ______ | (100%) |

There are a total of five problems.

1. (a) Compute the Gram–Schmidt QR-factorization of the matrix

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & -4 & 2 \end{bmatrix}. \qquad (15\%)$$

(b) Use the above to find the least square solution to

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{where} \quad \boldsymbol{b} = [2 \ \ 1 \ \ -1 \ \ 1]^T. \qquad (5\%)$$

Sol'n: (a) $\vec{a}_1 = \begin{bmatrix} -4 \\ 2 \\ 1 \\ 2 \end{bmatrix}$. $\vec{q}_1 = \frac{\vec{a}_1}{|\vec{a}_1|} = \frac{\vec{a}_1}{\sqrt{(-4)^2+2^2+1^2+2^2}} = \frac{\vec{a}_1}{\sqrt{25}} = \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix}$.

$$\vec{q}_2 = \frac{\vec{a}_2 - \langle \vec{a}_2, \vec{q}_1 \rangle \vec{q}_1}{|\vec{a}_2 - \langle \vec{a}_2, \vec{q}_1 \rangle|} = \left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \\ -4 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 0 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} \right\rangle \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} \right\} \Big/ \text{(its own length)}$$

$$= \left\{ \begin{bmatrix} 0 \\ 0 \\ -2 \\ -4 \end{bmatrix} - (-2/5 - 8/5) \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} \right\} \Big/ \text{(its own length)} = \begin{bmatrix} -8/5 \\ 4/5 \\ -8/5 \\ -16/5 \end{bmatrix} \Big/ \sqrt{(-\tfrac{8}{5})^2 + (\tfrac{4}{5})^2 + (-\tfrac{8}{5})^2 + (-\tfrac{16}{5})^2}$$

$$= \begin{bmatrix} -8/5 \\ 4/5 \\ -8/5 \\ -16/5 \end{bmatrix} \Big/ 4 = \begin{bmatrix} -2/5 \\ 1/5 \\ -2/5 \\ -4/5 \end{bmatrix}. \qquad \vec{q}_3 = \frac{\vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2}{|\vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2|} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} - \left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} \right\rangle \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} \right.$$

$$\left. - \left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 1/5 \\ -2/5 \\ -4/5 \end{bmatrix} \right\rangle \begin{bmatrix} -2/5 \\ 1/5 \\ -2/5 \\ -4/5 \end{bmatrix} \right\} \Big/ \text{(its own length)} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -4/5 \\ 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} - (-1) \begin{bmatrix} -2/5 \\ 1/5 \\ -2/5 \\ -4/5 \end{bmatrix} \right\} \Big/ \text{(its own length)}$$

$$= \begin{bmatrix} 2/5 \\ 4/5 \\ -8/5 \\ 4/5 \end{bmatrix} \div 2 = \begin{bmatrix} 1/5 \\ 2/5 \\ -4/5 \\ 2/5 \end{bmatrix}. \quad \text{Therefore } A = QR = \begin{bmatrix} -4/5 & -2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \\ 1/5 & -2/5 & -4/5 \\ 2/5 & -4/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5 & 4 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) The sol'n $\hat{x}$ is given by $R\hat{x} = Q^T \vec{b}$:

$$\begin{bmatrix} 5 & 4 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4/5 & 2/5 & 1/5 & 2/5 \\ -2/5 & 1/5 & -2/5 & -4/5 \\ 1/5 & 2/5 & -4/5 & 2/5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

From back substitution,

$2x_3 = 2, \quad x_3 = 1$

$2x_2 - x_3 = -1, \quad 2x_2 = 1 - 1 = 0, \quad x_2 = 0$

$5x_1 = -4x_2 - x_3 - 1 = -1 - 1 = -2, \quad x_1 = -2/5$

Hence $\hat{x} = \begin{bmatrix} -2/5 \\ 0 \\ 1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$. Find the diagonal matrix $D$ and a transformation matrix $S$ such that $D = S^{-1}AS$. (20%)

Sol'n (i) $\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)^2 + 1 = 0$

$\lambda = -3+i,\ -3-i$. These are the two eigenvalues

(ii) Find the eigenvector corresponding to $\lambda = -3+i$:

$$\left[\begin{array}{cc:c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{R_1 + iR_2} \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow[R_2 \to R_1]{R_1 \to R_2} \left[\begin{array}{cc:c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - i x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$. So $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is the eigenvector.

(iii) Find the eigenvector corresponding to $\lambda = -3-i$:

$$\left[\begin{array}{cc:c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \xrightarrow{R_1 - iR_2} \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right] \xrightarrow[R_2 \to R_1]{R_1 \to R_2} \left[\begin{array}{cc:c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + i x_2 = 0$,

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$. So $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is the eigenvector.

(iv)

$$D = \begin{bmatrix} -3+i & 0 \\ 0 & -3-i \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}.$$

3. (a) Expand the function $f(x) = x$ into a Fourier series on the interval $[0, L]$ by using the orthonormal basis

$$\left\{ \sqrt{\frac{1}{L}}, \sqrt{\frac{2}{L}} \cos\frac{k\pi x}{L} \;\middle|\; k = 1, 2, \ldots \right\}. \qquad (15\%)$$

(b) The set of functions $\{1, \cos kx, \sin kx \mid k = 1, 2, \ldots\}$ is known to be an *orthogonal* set on the interval $[-\pi, \pi]$. Explain how to make this set an *orthonormal* set. (5%)

Sol'n (a) $x = \sum_j \langle x, \varphi_j \rangle \varphi_j$.

$j = 0$: $\langle x, \sqrt{\frac{1}{L}} \rangle = \int_0^L x \cdot \sqrt{\frac{1}{L}}\, dx = \sqrt{\frac{1}{L}} \cdot \frac{1}{2} x^2 \Big|_0^L = \frac{L}{2}\sqrt{L}$;

$j > 0$: $\langle x, \sqrt{\frac{2}{L}} \cos\frac{j\pi x}{L} \rangle = \int_0^L x \cdot \sqrt{\frac{2}{L}} \cos\frac{j\pi x}{L}\, dx = \sqrt{\frac{2}{L}} \cdot \frac{L}{j\pi}\left[ x \sin\frac{j\pi x}{L} \Big|_{x=0}^{x=L} - \int_0^L \sin\frac{j\pi x}{L}\, dx \right]$

$$= \frac{1}{j\pi}\sqrt{2L} \cdot \frac{L}{j\pi}\left[ \cos\frac{j\pi x}{L} \Big|_{x=0}^{x=L} \right] = \frac{L\sqrt{2L}}{(j\pi)^2}\left[ (-1)^j - 1 \right].$$

Hence

$$x = \frac{L}{2}\sqrt{L} \cdot \sqrt{\frac{1}{L}} + \sum_{j=1}^{\infty} \frac{L\sqrt{2L}}{(j\pi)^2}\left[(-1)^j - 1\right] \cdot \sqrt{\frac{2}{L}} \cos\frac{j\pi x}{L}$$

$$= \frac{L}{2} + \sum_{k=1}^{\infty} \frac{4L}{(2k+1)^2\pi^2} \cos\left[\frac{(2k+1)\pi x}{L}\right].$$

(b) We must normalize each eigenvector:

(i) $\langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1 \cdot 1\, dx = 2\pi$. Thus $1$ is normalized to $\frac{1}{\sqrt{2\pi}} \cdot 1$;

(ii) $\langle \cos kx, \cos kx \rangle = \int_{-\pi}^{\pi} \cos kx \cdot \cos kx\, dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2kx)}{2}\, dx = \frac{1}{2} \cdot 2\pi = \pi$.

Thus $\cos kx$ is normalized to $\frac{1}{\sqrt{\pi}} \cos kx$.

(iii) $\langle \sin kx, \sin kx \rangle = \int_{-\pi}^{\pi} \sin kx \cdot \sin kx\, dx = \int_{-\pi}^{\pi} \frac{1 - \cos(2kx)}{2}\, dx = \frac{1}{2} \cdot 2\pi = \pi$.

Thus $\sin kx$ is normalized to $\frac{1}{\sqrt{\pi}} \sin kx$.

4. Given

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix},$$

find a matrix $S$ such that $J=S^{-1}AS$ is the Jordan canonical form of $A$. (20%)

<u>Sol'n</u> First, we determine all the eigenvalues:

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ 1 & -3 & 3-\lambda\end{vmatrix}=\lambda^2(3-\lambda)+1-3\lambda=-(\lambda^3-3\lambda^2+3\lambda-1)=-(\lambda-1)^3.$$

Thus $\lambda=1$.

Next, we determine the eigenvector and generalized eigenvectors.

$$\left[\begin{array}{ccc:c}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 1 & -3 & 2 & 0\end{array}\right]\begin{matrix}R_1\times(-1)\\ \\ R_3+R_1\end{matrix}\rightarrow\left[\begin{array}{ccc:c}1 & -1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & -2 & 2 & 0\end{array}\right]\begin{matrix}R_3-R_2\\ R_2\times(-1)\\ R_3-2R_2\end{matrix}\rightarrow\left[\begin{array}{ccc:c}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\begin{cases}x_1 \quad -x_3=0\\ \quad x_2-x_3=0\end{cases}\Rightarrow\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_3\\ x_3\\ x_3\end{bmatrix}=x_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}.$$ Thus $\vec{v}_1=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ is an eigenvector.

We now find a generalized eigenvector by

$$(A-\lambda I)\vec{v}_2=\vec{v}_1,$$

$$\left[\begin{array}{ccc:c}-1 & 1 & 0 & 1\\ 0 & -1 & 1 & 1\\ 1 & -3 & 2 & 1\end{array}\right]\begin{matrix}R_1\times(-1)\\ R_2\times(-1)\\ R_3+R_1\end{matrix}\rightarrow\left[\begin{array}{ccc:c}1 & -1 & 0 & -1\\ 0 & 1 & -1 & -1\\ 0 & -2 & 2 & 2\end{array}\right]\begin{matrix}R_1+R_2\\ \\ R_3+2R_2\end{matrix}\rightarrow\left[\begin{array}{ccc:c}1 & 0 & -1 & -2\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\begin{cases}x_1 \quad -x_3=-2\\ \quad x_2-x_3=-1\end{cases}\Rightarrow\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_3-2\\ x_3-1\\ x_3\end{bmatrix}=x_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}-2\\ -1\\ 0\end{bmatrix}.$$

Set $x_3=0$ in the above. We obtain a generalized eigenvector

$$\vec{v}_2=\begin{bmatrix}-2\\ -1\\ 0\end{bmatrix}.$$

Further, we find a second eigenvector by

$$(A-\lambda I)\vec{v}_3=\vec{v}_2,$$

$$\left[\begin{array}{ccc:c}-1 & 1 & 0 & -2\\ 0 & -1 & 1 & -1\\ 1 & -3 & 2 & 0\end{array}\right]\begin{matrix}R_1\times(-1)\\ R_2\times(-1)\\ R_3+R_1\end{matrix}\rightarrow\left[\begin{array}{ccc:c}1 & -1 & 0 & 2\\ 0 & 1 & -1 & 1\\ 0 & -2 & 2 & -2\end{array}\right]\begin{matrix}R_1+R_2\\ \\ R_3+2R_2\end{matrix}\rightarrow\left[\begin{array}{ccc:c}1 & 0 & -1 & 3\\ 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\begin{cases}x_1 \quad -x_3=3\\ \quad x_2-x_3=1\end{cases}\Rightarrow\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_3+3\\ x_3+1\\ x_3\end{bmatrix}=x_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}.$$ Setting $x_3=0$, we obtain $\vec{v}_3=\begin{bmatrix}3\\ 1\\ 0\end{bmatrix}$.

Therefore

$$J=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}=S^{-1}AS,\ \text{where}\ S=[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]=\begin{bmatrix}1 & -2 & 3\\ 1 & -1 & 1\\ 1 & 0 & 0\end{bmatrix}.$$

5. (a) Let

$$J = \begin{bmatrix} \lambda_0 & 1 & & & 0 \\ & \lambda_0 & 1 & & \\ & & \lambda_0 & \ddots & \\ & & & \ddots & 1 \\ & 0 & & & \lambda_0 \end{bmatrix}_{m\times m},$$

Give $e^{J(t-\tau)}$. (5%)

(b) Solve the system of linear differential equations

$$\begin{cases} \frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}; & \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\ \vec{x}(-3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

(15%)

Sol'n

(a)

$$e^{J(t-\tau)} = \begin{bmatrix} e^{\lambda_0(t-\tau)} & (t-\tau)e^{\lambda_0(t-\tau)} & \frac{(t-\tau)^2}{2!}e^{\lambda_0(t-\tau)} & \cdots & \frac{(t-\tau)^{m-1}}{(m-1)!}e^{\lambda_0(t-\tau)} \\ & e^{\lambda_0(t-\tau)} & (t-\tau)e^{\lambda_0(t-\tau)} & \frac{(t-\tau)^2}{2!}e^{\lambda_0(t-\tau)} & \vdots \\ & & \ddots & \ddots & \frac{(t-\tau)^2}{2!}e^{\lambda_0(t-\tau)} \\ & 0 & & \ddots & (t-\tau)e^{\lambda_0(t-\tau)} \\ & & & & e^{\lambda_0(t-\tau)} \end{bmatrix}.$$

(b) Find the eigenvalues of $\begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ by $\begin{vmatrix} 0-\lambda & 1 \\ -4 & 4-\lambda \end{vmatrix} = -\lambda(4-\lambda)+4 = \lambda^2-4\lambda+4 = (\lambda-2)^2 = 0$; so $\lambda=2$ has multiplicity 2. Next, find eigenvector and generalized eigenvector:

$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -2x_1+x_2=0,\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector.

$\left[\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -2x_1+x_2=1,\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1+1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; take $x_1=0$, getting $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a generalized eigenvector.

Hence

$$A \equiv \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\vec{x}(t) = e^{A(t+3)}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-3}^{t} e^{A(t-\tau)}\begin{bmatrix} 0 \\ 1 \end{bmatrix}d\tau = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{2(t+3)} & (t+3)e^{2(t+3)} \\ 0 & e^{2(t+3)} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \int_{-3}^{t} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{2(t-\tau)} & (t-\tau)e^{2(t-\tau)} \\ 0 & e^{2(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}d\tau$$

$$= -e^{2(t+3)}\begin{bmatrix} 2t+5 \\ 4t+12 \end{bmatrix} + \int_{-3}^{t} \begin{bmatrix} (t-\tau)e^{2(t-\tau)} \\ (2(t-\tau)+1)e^{2(t-\tau)} \end{bmatrix}d\tau$$