

There are a total of five problems. No calculators are allowed.

Name \_\_\_\_\_

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_
4. \_\_\_\_\_
5. \_\_\_\_\_

Total score \_\_\_\_\_

1. Let

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 5 & -2 \\ 4 & -1 & -9 \end{bmatrix}$$

- (i) Find an  $LU$ -factorization of  $A$  (i.e.,  $A = LU$ ) where  $L$  is unit lower triangular and  $U$  is upper triangular. (10%)
- (ii) Use part (i) to solve  $A\vec{x} = \vec{b}$ , where  $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ , with explanation. (8%)
- (iii) Does every square matrix admit an  $LU$ -factorization? Answer yes or no. (2%)

(i) We here apply only the elementary row operation of Type III:

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 5 & -2 \\ 4 & -1 & 9 \end{bmatrix} \begin{matrix} R_1 \\ R_2 - \frac{1}{2}R_1 \\ R_3 - 2R_1 \end{matrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & -1 \\ 0 & -9 & -5 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

$$\rightarrow = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & -1 \\ 0 & -9 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \text{ times}$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 8 \end{bmatrix}}_U, \text{ where } L \text{ is unit lower triangular, while } U \text{ is upper triangular.}$$

- (ii) Let  $U\vec{x} = \vec{z}$  and  $L\vec{z} = \vec{b}$ . Solve  $L\vec{z} = \vec{b}$  first by back substitution
- $$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \rightarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ -1/2 \end{bmatrix}$$
- Then solve  $U\vec{x} = \vec{z}$  by forward substitution
- $$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 3 & -1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ -1/2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -23/48 \\ 25/48 \\ 1/16 \end{bmatrix}$$

(iii) No.

Name \_\_\_\_\_

2. Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{bmatrix}$$

by using the Gauss-Jordan elimination to reach a reduced row echelon form. (No credits if other methods are used.) (20%)

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & -7 & -1 & 1 & 0 \\ 0 & -2 & -8 & -2 & 0 & 1 \end{array} \right] \begin{array}{l} \\ R_2 \times (-1) \\ R_3 - 2R_2 \end{array}$$

$$\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 7 & 1 & -1 & 0 \\ 0 & 0 & 6 & 0 & -2 & 1 \end{array} \right] \begin{array}{l} R_1 - \frac{5}{6}R_3 \\ R_2 - \frac{7}{6}R_3 \\ R_3 \div 6 \end{array} \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & \frac{5}{3} & -\frac{5}{6} \\ 0 & 1 & 0 & 1 & \frac{4}{3} & -\frac{7}{6} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{6} \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ \\ \end{array}$$

$$\rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & \frac{3}{2} \\ 0 & 1 & 0 & 1 & \frac{4}{3} & -\frac{7}{6} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{6} \end{array} \right]$$

So

$$A^{-1} = \begin{bmatrix} -1 & -1 & \frac{3}{2} \\ 1 & \frac{4}{3} & -\frac{7}{6} \\ 0 & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

Name \_\_\_\_\_

3. Solve the following system of linear equations by reducing it to reduced row echelon form. Your solutions should be represented as a column vector where each free variable is separated from the other free variables.

$$\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 - 4x_5 + 13x_6 & = -3, \\ x_1 + 3x_2 - x_3 + 3x_4 - 2x_5 + 16x_6 & = -4, \\ x_1 - x_3 + 3x_4 - 8x_5 + 7x_6 & = -1, \\ x_1 + 3x_2 - x_3 + 3x_4 - 2x_5 + 16x_6 + 4x_7 & = 4, \\ 2x_1 + 3x_2 - 2x_3 + 6x_4 - 8x_5 + 23x_6 + 4x_7 & = 3, \\ x_4 & = 1 \end{cases} \quad (25\%)$$

$$\left[ \begin{array}{cccccccc|c} 1 & 2 & -1 & 3 & -4 & 13 & 0 & -3 \\ 1 & 3 & -1 & 3 & -2 & 16 & 0 & -4 \\ 1 & 0 & -1 & 3 & -8 & 7 & 0 & -1 \\ 1 & 3 & -1 & 3 & -2 & 16 & 4 & 4 \\ 2 & 3 & -2 & 6 & -8 & 23 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - 2R_1 \end{array} \rightarrow \left[ \begin{array}{cccccccc|c} 1 & 2 & -1 & 3 & -4 & 13 & 0 & -3 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & -1 \\ 0 & -2 & 0 & 0 & -4 & -6 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 3 & 4 & 7 \\ 0 & -1 & 0 & 0 & 0 & -3 & 4 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 - 2R_2 \\ R_2 \\ R_3 + 2R_2 \\ R_4 - R_2 \\ R_5 + R_2 \end{array}$$

$$\rightarrow \left[ \begin{array}{cccccccc|c} 1 & 0 & -1 & 3 & -8 & 7 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 2 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_6 \\ R_4 \div 4 \\ R_5 \div 2 \\ R_6 \rightarrow R_3 \end{array} \rightarrow \left[ \begin{array}{cccccccc|c} 1 & 0 & -1 & 3 & -8 & 7 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 - 3R_3 \\ R_4 \rightarrow R_5 \\ R_5 \rightarrow R_4 \end{array}$$

$$\rightarrow \left[ \begin{array}{cccccccc|c} 1 & 0 & -1 & 0 & -8 & 7 & 0 & -4 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] R_4 - 2R_5 \rightarrow \left[ \begin{array}{cccccccc|c} 1 & 0 & -1 & 0 & -8 & 7 & 0 & -4 \\ 0 & 1 & 0 & 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 + 8R_4 \\ R_2 - 2R_4 \end{array}$$

$$\rightarrow \left[ \begin{array}{cccccccc|c} 1 & 0 & -1 & 0 & 0 & 7 & 0 & -4 \\ 0 & 1 & 0 & 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_3 - 7x_6 - 4 \\ -3x_6 - 1 \\ x_3 \\ 1 \\ 0 \\ x_6 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Name \_\_\_\_\_

4. Let  $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & -1 \\ -1 & 1 & 2 & 1 \end{bmatrix}$ . Use the Gauss-Jordan reduction to determine

(i)  $N(A)$ .

( $N(A)$  is the nullspace of  $A$ )

(7.5%)

(ii)  $R(A)$ .

(7.5%)

(i) and (ii) can be considered together from  $A\vec{x} = \vec{b}$ , where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}:$$

$$\begin{aligned} & \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & b_1 \\ 3 & 1 & 0 & -1 & b_2 \\ -1 & 1 & 2 & 1 & b_3 \end{array} \right] \begin{array}{l} R_1 \\ R_2 - 3R_1 \\ R_3 + R_1 \end{array} \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & b_1 \\ 0 & -2 & -3 & -1 & b_2 - 3b_1 \\ 0 & 2 & 3 & 1 & b_3 + b_1 \end{array} \right] \begin{array}{l} \\ R_2 \\ R_3 + R_2 \end{array} \\ & \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & b_1 \\ 0 & -2 & -3 & -1 & b_2 - 3b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array} \right] \begin{array}{l} \\ R_2 \div (-2) \\ \end{array} \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & b_1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2}b_2 + \frac{3}{2}b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array} \right] \begin{array}{l} \\ R_1 - R_2 \\ \end{array} \\ & \rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}b_1 + \frac{1}{2}b_2 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2}b_2 + \frac{3}{2}b_1 \\ 0 & 0 & 0 & 0 & -2b_1 + b_2 + b_3 \end{array} \right] \end{aligned}$$

For the above system to be consistent, we must have

$$-2b_1 + b_2 + b_3 = 0, \text{ i.e., } b_3 = 2b_1 - b_2$$

thus

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 - b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

This gives

$$R(A) = \left\{ \vec{b} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \mid b_1, b_2 \in \mathbb{R} \text{ are arbitrary} \right\} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

Set  $\vec{b} = \vec{0}$ . Then  $(x_1, x_2, x_3, x_4)$  satisfy  $\left[ \begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Hence

$$\begin{aligned} N(A) &= \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ -\frac{3}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R} \text{ are arbitrary} \right\} \\ &= \text{span} \left( \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right). \end{aligned}$$

Name \_\_\_\_\_

5. (i) State and prove Cramer's Rule.

(12%)

(ii) Use Cramer's Rule to solve

$$\begin{cases} x_1 + 2x_2 + x_3 = 5, \\ 2x_1 + 2x_2 + x_3 = 6, \\ x_1 + 2x_2 + 3x_3 = 9. \end{cases} \quad (8\%)$$

(i) Statement Let  $A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$  be an  $n \times n$  nonsingular matrix with column vectors  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ . For given  $\vec{b} \in \mathbb{R}^n$ , define

$$A_j = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{j-1}, \vec{b}, \vec{a}_{j+1}, \dots, \vec{a}_n].$$

Then the solution  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  of  $A\vec{x} = \vec{b}$  is given by

$$x_j = \frac{\det A_j}{\det A}, \quad \text{for } j=1, 2, \dots, n. \quad (*)$$

Proof: We know that for a nonsingular matrix  $A$ ,

$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$

Thus

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{\det A} (\text{adj } A)\vec{b}.$$

The  $x_j$ -component of  $\vec{x}$  is, therefore,

$$x_j = \frac{1}{\det A} \sum_{k=1}^n A_{kj} b_k, \quad \text{where } A_{ji} \text{ is the } (j,i)\text{-cofactor of the matrix } A.$$

But  $\sum_{k=1}^n A_{kj} b_k = \det A_j$ . Therefore (\*) has been proved.  $\square$

(ii)

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 6 + 4 + 2 - 2 - 12 - 2 = -4$$

$$x_1 = \left(-\frac{1}{4}\right) \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = 1;$$

$$x_2 = \left(-\frac{1}{4}\right) \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = 1;$$

$$x_3 = \left(-\frac{1}{4}\right) \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = 2.$$