MATH 601 Section 603 Final Exam Fall 2004 (12/8/2004)

Name ____________________
E-mail (print) ____________________
(if you wish to receive grades by e-mail)

1.	(12.5%)	5.		(12.5%)	
2.	(12.5%)	6.		(12.5%)	
3.	(12%)	7.		(13%)	
4.	(13%)	8.		(12%)	
Total					

There are a total of eight problems.

1. Let S be the paraboloid $z = 16 - (x^2 + y^2)$ lying above the xy-plane. Let

$$\boldsymbol{F} = (-3z + x^2y + 3y)\boldsymbol{i} + (2x + 5z)\boldsymbol{j} + (3x - 27)\boldsymbol{k}.$$

Show that

$$\iint_S \nabla \times \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_C \vec{F} \cdot d\vec{r}$$

where C is the bounding curve of S (on the xy-plane). (12.5)%
(Hint: $\cos^2\theta \sin^2\theta = \frac{1}{8}(1 - \cos 4\theta)$)

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3z+x^2y+3y & 2x+5z & 3x-27 \end{vmatrix} = \vec{i}(-5)+\vec{j}(-3-3)+\vec{k}(2-x^2-3) = -5\vec{i}-6\vec{j}-(x^2+1)\vec{k}$$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_{D_{xy}} \left[-5\vec{i}-6\vec{j}-(x^2+1)\vec{k}\right]\cdot\left[2x\vec{i}+2y\vec{j}+\vec{k}\right]dx\,dy$$

$$= \iint_{D_{xy}} \left[-10x-12y-(x^2+1)\right]dx\,dy = \int_0^{2\pi}\int_0^4 \left[-10(r\cos\theta)-12(r\sin\theta)-(r^2\cos^2\theta+1)\right]r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^4\left[r^2\cdot\frac{1+\cos 2\theta}{2}+1\right]r\,dr\,d\theta = -2\pi\cdot\left[\frac{r^4}{8}+\frac{r^2}{2}\right]\Bigg|_{r=0}^{r=4} = -80\pi$$

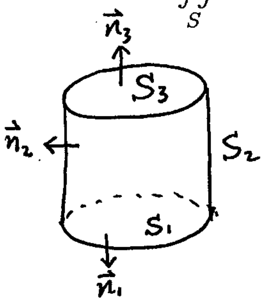
$$\int_C \vec{F}\cdot d\vec{r} = \int_C (-3z+x^2y+3y)dx+(2x+5z)dy+(3x-27)dz$$

$x = 4\cos\theta$
$y = 4\sin\theta$
$z = 0$

$$= \int_0^{2\pi}\left\{\left[16\cos^2\theta\cdot 4\sin\theta+12\sin\theta\right]d(4\cos\theta)+(8\cos\theta)\,d(4\sin\theta)\right\}$$

$$= \int_0^{2\pi}\left\{-256\cos^2\theta\sin^2\theta-48\sin^2\theta+32\cos^2\theta\right\}d\theta = (-32-8)\cdot 2\pi = -80\pi.$$

2. Let S be the closed cylinder bounded laterally by $x^2+y^2=9$, on the bottom by $z=0$ and on the top by $z=6$. Let $\boldsymbol{F} = x^3\boldsymbol{i} + y^3\boldsymbol{j} - (z-5)\boldsymbol{k}$. Evaluate the surface integral $\displaystyle\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, where the surface normal of S is oriented outward. (12.5%)



S_1: bottom surface $z=0$, $\vec{n}_1 = -\vec{k}$,

S_2: lateral surface, $\vec{n}_2 = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$,

S_3: top surface $z=6$, $\vec{n}_3 = \vec{k}$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} \cdots$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D_{xy}} \left[x^3\vec{i} + y^3\vec{j} - (z-5)\vec{k}\right]\cdot\left[-\vec{k}\right] dx\,dy = \iint_{D_{xy}} (\overset{0}{z}-5)\,dx\,dy$$

$$= -5\iint_{D_{xy}} dx\,dy = -5\pi(3)^2 = -45\pi.$$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \int_0^6\!\!\int_0^{2\pi} \left[x^3\vec{i} + y^3\vec{j} - (z-5)\vec{k}\right]\cdot \underbrace{\left[\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right]}_{\vec{n}} \underbrace{3\,d\theta\,dz}_{dS} \qquad \begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \end{cases}$$

$$= 3\int_0^6\!\!\int_0^{2\pi} \left[27\cos^3\theta\cdot\cos\theta + 27\sin^3\theta\cdot\sin\theta\right] d\theta\,dz$$

$$= 3\cdot 6\cdot 27\int_0^{2\pi}\left[\cos^4\theta + \sin^4\theta\right] d\theta$$

$$\boxed{\begin{aligned} &\cos^4\theta + \sin^4\theta \\ &= (\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta \\ &= 1 - 2\cdot\tfrac{1}{8}(1-\cos 4\theta) \\ &= \tfrac{3}{4} + \tfrac{1}{4}\cos 4\theta \end{aligned}}$$

$$= 486\cdot\frac{3}{4}\cdot 2\pi = 729\pi$$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_{D_{xy}} \left[x^3\vec{i} + y^3\vec{j} - (z-5)\vec{k}\right]\cdot\vec{k}\,dx\,dy = -\iint_{D_{xy}} (\overset{6}{z}-5)\,dx\,dy$$

$$= -\iint_{D_{xy}} dx\,dy = -\pi(3)^2 = -9\pi$$

$$\iint_S \vec{F}\cdot d\vec{S} = -45\pi + 729\pi - 9\pi = \underline{\underline{675\pi}}$$

3. Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $r = |\boldsymbol{r}|$, $f = f(x,y,z)$, $g = g(x,y,z)$

$$\boldsymbol{F} = \boldsymbol{F}(x,y,z) = P_1(x,y,z)\boldsymbol{i} + Q_1(x,y,z)\boldsymbol{j} + R_1(x,y,z)\boldsymbol{k},$$
$$\boldsymbol{G} = \boldsymbol{G}(x,y,z) = P_2(x,y,z)\boldsymbol{i} + Q_2(x,y,z)\boldsymbol{j} + R_2(x,y,x)\boldsymbol{k}.$$

Derive the following identities:

(a) $\nabla \cdot (r^n \boldsymbol{r}) = (n+3)r^n$ for any real number n; (4%)

(b) $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot \nabla \times \boldsymbol{F} - \boldsymbol{F} \cdot \nabla \times \boldsymbol{G}$; (4%)

(c) $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$. (4%)

Sol'n: (a) $\nabla \cdot (r^n \vec{r})$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) = \left(r^n + x\frac{\partial}{\partial x}r^n\right) + \left(r^n + y\frac{\partial}{\partial y}r^n\right) + \left(r^n + z\frac{\partial}{\partial z}r^n\right)$$

$$= \left(r^n + x \cdot n r^{n-1}\frac{x}{r}\right) + \left(r^n + y \cdot n r^{n-1}\frac{y}{r}\right) + \left(r^n + z \cdot n r^{n-1}\frac{z}{r}\right)$$

$$= 3r^n + n r^{n-1}\frac{x^2+y^2+z^2}{r} = 3r^n + n r^{n-1} r = (n+3)r^n.$$

4. Let D be the solid bounded by the following three surfaces:

lateral: $x^2 + y^2 = a^2$

top: $z = b_1$,

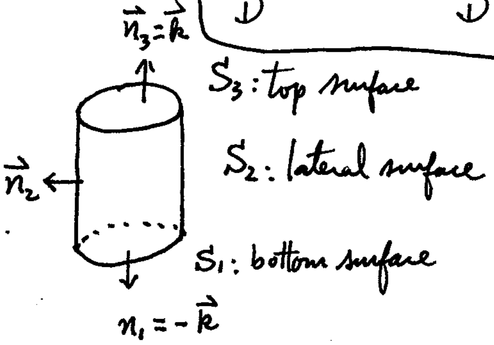
bottom: $z = b_2$, where $b_1 > b_2$.

Let $\boldsymbol{F} = 2x\boldsymbol{i} - 3y\boldsymbol{j} - (11z - 9)\boldsymbol{k}$. Verify the Gauss Divergence Theorem for the given $\boldsymbol{F}, D$ and ∂D (∂D is the bounding surface of D). (13%)

Gauss Theorem says $\iiint_D \nabla\cdot\vec{F}\,dV = \iint_{\partial D} \vec{F}\cdot d\vec{S}$.

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3y) + \frac{\partial}{\partial z}(-11z) = 2-3-11 = -12.$$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D (-12)\,dV = -12\cdot\text{volume of } D = (-12)\,\pi a^2(b_1-b_2).$$



$$\iint_{\partial D} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\vec{n}_1\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}_2\,dS + \iint_{S_3} \vec{F}\cdot\vec{n}_3\,dS$$

$$= \iint_{D_{xy}} \left[2x\vec{i} - 3y\vec{j} - (11z-9)\vec{k}\right]\cdot\vec{n}_1\,dx\,dy + \int_{b_2}^{b_1}\int_0^{2\pi} \underbrace{\left[2x\vec{i} - 3y\vec{j} - (11z-9)\vec{k}\right]}_{\boxed{\begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \end{array}}}\cdot\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right) a\,d\theta\,dz$$

$$+ \iint_{D_{xy}} \left[2x\vec{i} - 3y\vec{j} - (11z-9)\right]\cdot\vec{n}_3\,dx\,dy$$

$$= \iint_{D_{xy}} (11\underset{\overset{\updownarrow}{b_2}}{z} - 9)\,dx\,dy + \int_{b_2}^{b_1}\int_0^{2\pi} \left[2\cdot a\cos\theta\cdot\cos\theta - 3\,a\sin\theta\cdot\sin\theta\right] a\,d\theta\,dz - \iint_{D_{xy}} (11\underset{\overset{\updownarrow}{b_1}}{z} - 9)\,dx\,dy$$

$$= (11b_2 - 9)\cdot\pi a^2 + (b_1-b_2)a^2\int_0^{2\pi} \left[2\cos^2\theta - 3\sin^2\theta\right]d\theta - (11b_1-9)\pi a^2$$

$$= \pi a^2\left[(11b_2-9) - (b_1-b_2) - (11b_1-9)\right] = -12\pi a^2(b_1-b_2).$$

5. (a) Compute the Gram–Schmidt QR-factorization of the matrix

$$A = \begin{bmatrix} -1 & 2 & 1 \\ -4 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \end{bmatrix}. \qquad (8.5\%)$$

(b) Use the above to find the least square solution to

$$A\boldsymbol{x} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} = [1 \ 2 \ 1 \ 1]^T. \qquad (4\%)$$

Sol'n: (a) $\vec{q}_1 = \dfrac{\vec{a}_1}{|\vec{a}_1|} = (-1,-4,2,2)\Big/\sqrt{(-1)^2+(-4)^2+2^2+2^2} = (-\frac{1}{5}, -\frac{4}{5}, \frac{2}{5}, \frac{2}{5})$

$$\vec{q}_2 = \frac{\vec{a}_2 - \langle \vec{a}_2, \vec{q}_1 \rangle \vec{q}_1}{|\vec{a}_2 - \langle \vec{a}_2, \vec{q}_1 \rangle \vec{q}_1|} = \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ -4 \end{bmatrix} - \left(-\frac{2}{5} - \frac{8}{5}\right) \begin{bmatrix} -\frac{1}{5} \\ -\frac{4}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{bmatrix} \right) \Big/ \text{its own length}$$

$$= \begin{bmatrix} 8/5 \\ -8/5 \\ 4/5 \\ -16/5 \end{bmatrix} \div \underbrace{\sqrt{(\tfrac{8}{5})^2 + (-\tfrac{8}{5})^2 + (\tfrac{4}{5})^2 + (-\tfrac{16}{5})^2}}_{4} = \begin{bmatrix} 2/5 \\ -2/5 \\ 1/5 \\ -4/5 \end{bmatrix}$$

$$\vec{q}_3 = \frac{\vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2}{|\vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2|} = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \underbrace{\left(-\frac{1}{5} + \frac{2}{5} + \frac{4}{5}\right)}_{1} \begin{bmatrix} -1/5 \\ -4/5 \\ 2/5 \\ 2/5 \end{bmatrix} - \underbrace{\left(\frac{2}{5} + \frac{1}{5} - \frac{8}{5}\right)}_{-1} \begin{bmatrix} 2/5 \\ -2/5 \\ 1/5 \\ -4/5 \end{bmatrix} \right) \Big/ \text{its own length}$$

$$= \begin{bmatrix} 8/5 \\ 2/5 \\ 4/5 \\ 4/5 \end{bmatrix} \div 2 = \begin{bmatrix} 4/5 \\ 1/5 \\ 2/5 \\ 2/5 \end{bmatrix}.$$

Hence

$$A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3] = QR = [\vec{q}_1 \ \vec{q}_2 \ \vec{q}_3] \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ -\frac{4}{5} & -\frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{4}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(b) The least square sol'n $\hat{x}$ is given by

$$R\hat{x} = Q^T\vec{b}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} -1/5 & -4/5 & 2/5 & 2/5 \\ 2/5 & -2/5 & 1/5 & -4/5 \\ 4/5 & 1/5 & 2/5 & 2/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

Back substitution yields

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 0 \\ 1 \end{bmatrix}.$$

6. (a) Let

$$J = \begin{bmatrix} \lambda_0 & 1 & 0 & 0 & \cdots & 0 \\ & \lambda_0 & 1 & 0 & \cdots & 0 \\ & & \lambda_0 & 1 & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & \bigcirc & & \ddots & 1 \\ & & & & & \lambda_0 \end{bmatrix}. \text{ Give } \quad e^{J(t-\lambda)}.$$

(4%)

(b) Solve the following system of linear differential equations

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}(t) = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \boldsymbol{x}(3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{cases}$$

(8.5%)

Sol'n

(a)

$$e^{J(t-\lambda)} = \begin{bmatrix} e^{\lambda_0(t-\lambda)} & (t-\lambda)e^{\lambda_0(t-\lambda)} & \frac{(t-\lambda)^2}{2!}e^{\lambda_0(t-\lambda)} & \cdots & \frac{(t-\lambda)^{m-1}}{(m-1)!}e^{\lambda_0(t-\lambda)} \\ & e^{\lambda_0(t-\lambda)} & (t-\lambda)e^{\lambda_0(t-\lambda)} & & \vdots \\ & & e^{\lambda_0(t-\lambda)} & \ddots & \vdots \\ & \bigcirc & & \ddots & (t-\lambda)e^{\lambda_0(t-\lambda)} \\ & & & & e^{\lambda_0(t-\lambda)} \end{bmatrix}$$

(b)

$$\begin{vmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{vmatrix} = -\lambda(6-\lambda)+9 = \lambda^2-6\lambda+9 = (\lambda-3)^2, \quad \lambda = 3$$

$$\left[\begin{array}{cc|c} 0-3 & 1 & 0 \\ -9 & 6-3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3 & 1 & 0 \\ -9 & 3 & 0 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad -3x_1+x_2=0, \ x_2=3x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} -3 & 1 & 1 \\ -9 & 3 & 3 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} -3 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad -3x_1+x_2=1, \ x_2=1+3x_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 1+3x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad A = S\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}S^{-1}$$

$$\vec{x}(t) = e^{A(t-3)}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_3^t e^{A(t-\tau)}\begin{bmatrix} 0 \\ 1 \end{bmatrix}d\tau = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} e^{3(t-3)} & (t-3)e^{3(t-3)} \\ 0 & e^{3(t-3)} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} + \int_3^t \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} e^{3(t-\tau)} & (t-\tau)e^{3(t-\tau)} \\ 0 & e^{3(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}d\tau$$

$$= \begin{bmatrix} e^{3(t-3)} - 3(t-3)e^{3(t-3)} \\ -9(t-3)e^{3(t-3)} \end{bmatrix} + \int_3^t \begin{bmatrix} (t-\tau)e^{3(t-\tau)} \\ 3(t-\tau)e^{3(t-\tau)} + e^{3(t-\tau)} \end{bmatrix}d\tau$$

7. Let

$$A = \begin{bmatrix} 1 & -5 & 2 \\ 2 & -4 & 2 \\ 4 & -1 & -9 \end{bmatrix}.$$

(a) Find an LU-factorization of A (i.e., $A = LU$) where L is unit lower triangular and U is upper triangular. (9%)

(b) Use part (a) to solve $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}$. (4%)

Sol'n

(a) $\begin{bmatrix} 1 & -5 & 2 \\ 2 & -4 & 2 \\ 4 & -1 & -9 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-4R_1 \end{matrix} \longrightarrow \begin{bmatrix} 1 & -5 & 2 \\ 0 & 6 & -2 \\ 0 & 19 & -17 \end{bmatrix} R_3 - \frac{19}{6}R_2 \longrightarrow \begin{bmatrix} 1 & -5 & 2 \\ 0 & 6 & -2 \\ 0 & 0 & -\frac{32}{3} \end{bmatrix}$

Thus

$$A = \begin{bmatrix} 1 & -5 & 2 \\ 2 & -4 & 2 \\ 4 & -1 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & \frac{19}{6} & 1 \end{bmatrix} \begin{bmatrix} 1 & -5 & 2 \\ 0 & 6 & -2 \\ 0 & 0 & -\frac{32}{3} \end{bmatrix} = LU.$$

(b) $LU\vec{x} = \vec{b}$, $U\vec{x} = \vec{z}$, $L\vec{z} = \vec{b}$

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & \frac{19}{6} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}$,

$z_1 = -2$;

$z_2 = -2z_1 - 1 = 4 - 1 = 3$;

$z_3 = -4z_1 - \frac{19}{6}z_2 - 3 = 8 - \frac{57}{6} - 3 = -\frac{27}{6} = -\frac{9}{2}$

$\begin{bmatrix} 1 & -5 & 2 \\ 0 & 6 & -2 \\ 0 & 0 & -\frac{32}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -\frac{9}{2} \end{bmatrix} \Rightarrow$

$x_3 = \frac{3}{32} \cdot \frac{9}{2} = \frac{27}{64}$

$x_2 = \frac{1}{6}(2x_3 + 3) = \frac{1}{6}\left(\frac{27}{32} + 3\right) = \frac{41}{64}$

$x_1 = 5x_2 - 2x_3 - 2 = 5 \cdot \frac{41}{64} - 2 \cdot \frac{27}{64} - 2 = \frac{23}{64}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 23/64 \\ 41/64 \\ 27/64 \end{bmatrix}.$$

8. Let A be an $m \times n$ matrix. Show that

(a) If $x \in N(A^TA)$, then $Ax \in R(A) \cap N(A^T)$. (3%)

(b) $N(A^TA) = N(A)$. (3%)

(c) A and A^TA have the same rank. (3%)

(d) If A has linearly independent columns, than A^TA is nonsingular. (3%)

Sol'n:

(a) $\vec{x} \in N(A^TA) \Rightarrow A^T(A\vec{x}) = \vec{0} \Rightarrow A\vec{x} \in N(A^T)$. Also, $A\vec{x} \in R(A)$ because $A\vec{x} = A\vec{y}$ for some $\vec{y} \in \mathbb{R}^m$ because we can just choose $\vec{y} = \vec{x}$.

(b) First, $N(A) \subseteq N(A^TA)$, as for any $\vec{x} \in N(A)$, $A\vec{x} = \vec{0}$ and hence $A^T(A\vec{x}) = \vec{0} = (A^TA)(\vec{x})$.

Next, $N(A^TA) \subseteq N(A)$, as for $\vec{x} \in N(A^TA)$, $(A^TA)\vec{x} = \vec{0}$.

Thus $\langle (A^TA)\vec{x}, \vec{x} \rangle = 0 = \langle A^T(A\vec{x}), \vec{x} \rangle = \langle A\vec{x}, A\vec{x} \rangle = |A\vec{x}|^2$. This forces $A\vec{x} = \vec{0}$ and hence $\vec{x} \in N(A)$.

(c) A's size is $m \times n$ while A^TA's size is $n \times n$. So

rank A + nullity $A = n$, rank A^TA + nullity $A^TA = n$.

But by part (b), nullity A = nullity A^TA.

Hence rank A = rank A^TA.

(d) If A has columns all linearly indep., then rank $A = n$. By part (c), $N(A^TA) = \{\vec{0}\}$. Thus, $A^TA\vec{x} = \vec{0}$ has no nontrivial solutions $\vec{x}$. Hence A^TA is nonsingular.