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On minimal cubature formulae for product weight functions[1]

Borislav Bojanov[a], Guergana Petrova[*,b]

[a] Department of Mathematics, University of Sofia, Blvd. James Boucher 5, 1126 Sofia, Bulgaria

[b] Department of Mathematics, University of South Carolina, Columbia, SC 29208, United States

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Abstract

We derive in a simple way certain minimal cubature formulae, obtained by Morrow and Patterson [2], and Xu [4], using a different technique. We also obtain in explicit form new near minimal cubature formulae. Then, as a corollary, we get a compact expression for the bivariate Lagrange interpolation polynomials, based on the nodes of the cubature.

Keywords: Numerical integration; Minimal and near-minimal cubature formulae; Lagrange interpolation

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1. Introduction

As usual, we denote by

$$\pi_n(\mathbb{R}^2) := \left\{ \sum_{i+j\leqslant n} a_{ij} x^i y^j \right\},$$

the set of all algebraic polynomials of total degree n in two variables.

Let $w(x)$ be a given weight function on $[-1,1]$. Define the product weight on $[-1,1]^2$ as

$$W^{(2)}(x,y) := w(x)w(y), \quad (x,y)\in[-1,1]^2.$$

We shall deal with cubature formulae of the form

$$I(f) := \int\int_{[-1,1]^2} W^{(2)}(x,y) f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \sum_{j=1}^{N} A_j f(x_j,y_j),$$

* Corresponding author. E-mail: petrova@math.sc.edu.

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where $(x_i, y_i) \in [-1,1]^2$ and $\{A_i\}$ are real coefficients. There have been many efforts to construct a formula with fixed number of nodes that integrates algebraic polynomials of degree as high as possible, or equivalently, given a fixed degree $2n-1$, to find a cubature with minimal number $N^*(n)$ of nodes that integrates exactly all polynomials from $\pi_{2n-1}(\mathbb{R}^2)$. Such a formula is called *minimal.* Möller [1] has shown that for centrally symmetric weight functions $N^*(n) \geqslant \frac{1}{2}(n(n+1)) + [\frac{1}{2}n]$. Recently, Xu [4] derived in explicit form certain minimal and near minimal formulae for the product Tchebycheff weight $W_0^{(2)}(x,y) := 1/(\sqrt{1-x^2})\,1/(\sqrt{1-y^2})$. The aim of this note is to demonstrate a simple approach that yields as an immediate application the formulae obtained in [4, 2] in a quite different way. We use these minimal (or near minimal) formulae to derive the d-dimensional cubatures, listed in Theorem 2.3, that are exact for all polynomials of degree $2n-1$ in d variables. Also, following a simple technique as in the univariate case, we get a Lagrange interpolation formula for any $P(x,y) \in \pi_{2n-1}(\mathbb{R}^2)$. Xu obtained this formula in [4], using different tools.

2. Results

Let us first introduce some notations.

Consider any fixed quadrature formula on $[-1,1]$, namely,

$$I'(f) := \int_{-1}^{1} w(t) f(t)\,\mathrm{d}t \approx \sum_{j=1}^{s} a_j f(x_j) =: L(f) \tag{2.1}$$

with nodes $x_1 < \cdots < x_s$ in $[-1,1]$. Set

$$\mathrm{O}(f) := \sum_{\mathrm{o}} a_j f(x_j), \qquad E(f) := \sum_{\mathrm{e}} a_j f(x_j),$$

where the subscript "o" (respectively, "e") indicates that the summation is expanded over the odd (even) indeces. Clearly, $L(f) = E(f) + \mathrm{O}(f)$. Sometimes we shall write $E_x(f)$ to denote that the operator E is applied to the function $f(x,y)$ with respect to the variable marked.

Let $\{P_k\}$ be the sequence of univariate orthogonal polynomials, associated with the weight $w(x)$ on $[-1,1]$. Finally, we set

$$A(f) := (\mathrm{O}_x\mathrm{O}_y + E_xE_y)(f), \qquad B(f) := (\mathrm{O}_xE_y + E_x\mathrm{O}_y)(f).$$

Theorem 2.1. *Assume that the quadrature formula* (2.1) *is exact for all polynomials of degree* $2n-1$ *and satisfies the conditions*

$$\sum_{\mathrm{o}} a_j = \sum_{\mathrm{e}} a_j, \tag{2.2}$$

$$E(P_k) = 0 \text{ or } \mathrm{O}(P_k) = 0 \quad \textit{for } k = 1, \ldots, n-1. \tag{2.3}$$

Then the cubatures

$$I(f) \approx 2A(f)$$

and

$$I(f) \approx 2B(f)$$

are exact for every $f \in \pi_{2n-1}(\mathbb{R}^2)$.

Next, we apply this rather elementary result based on rule (2.1) to get the cubatures, established by Xu [4] and Morrow and Patterson [2]. Let $w_0(x) := 1/(\sqrt{1-x^2})$. Then the following statements hold.

Corollary 2.2. *For each natural number n the cubature formulae*

$$\int\int_{[-1,1]^2} w_0(x)w_0(y)f(x,y)\,dx\,dy \approx 2B(f) \tag{2.4}$$

and

$$\int\int_{[-1,1]^2} w_0(x)w_0(y)f(x,y)\,dx\,dy \approx 2A(f) \tag{2.5}$$

are exact for all polynomials $f \in \pi_{2n-1}(\mathbb{R}^2)$. *Moreover,* (2.4) *is minimal for even n and near minimal for odd n* (2.5) *is near minimal for all n.*

Cubature formulae over the cube $B^d := [-1,1]^d$ in $\mathbb{R}^d$, $d > 2$ can be obtained by a repeat application of Theorem 2.1. We concentrate here only on the formulae, corresponding to the product Tchebycheff weight

$$W_0^{(d)}(x_1,\ldots,x_d) := w_0(x_1)\cdots w_0(x_d).$$

Denote $\boldsymbol{x}_d := \boldsymbol{x} := (x_1,\ldots,x_d)$. We shall use the summation operators $L_i(f)$ (or $A_{ij}(f)$), defined obviously as L (or A) acting on the function $f(x_1,\ldots,x_d)$, considered as a function of x_i (or x_i and x_j, respectively). Then the following theorem is true.

Theorem 2.3. *The cubature formulae*

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,d\boldsymbol{x} \approx 2^v A_{12}A_{34}\cdots A_{d-1,d}(f) \quad \text{for } d = 2v$$

and

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,d\boldsymbol{x} \approx 2^v A_{12}A_{34}\cdots A_{d-2,d-1}L(f) \quad \text{for } d = 2v+1,$$

or

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,d\boldsymbol{x} \approx 2^v A_{12}\cdots A_{2s-1,2s}L_{2s+1}A_{2s+2,2s+3}\cdots A_{d-1,d}(f) \quad \text{for } d = 2v+1$$

are exact for all polynomials of degree $2n-1$ *in d variables.*

Note that the first formula uses $(2m^2+2m)^v$ nodes for $n = 2m$ and $(2m^2)^v$ nodes for $n = 2m-1$. The number of the nodes for the second and the third one are $(2m+1)(2m^2+2m)^v$, when $n = 2m$ and $2m(2m^2)^v$, when $n = 2m-1$. ($L(f)$ uses $n+1$ nodes.)

Once we derive cubature formulae of high degree of precision, we can get explicit expressions for the fundamental polynomials for the Lagrange-type interpolation. It turns out that a simple approach can be applied to the case of cubatures, associated with the product Tchebycheff weight and we obtain the next interpolation formulae.

Theorem 2.4. *Every $P(x,y)\in\pi_{n-1}(\mathbb{R}^2)$ can be written as*

$$\begin{aligned}P(x,y)=&\frac{1}{n^2}\sum_{i=0}^{m-1}(P(\eta_0^{(n)},\eta_{2i+1}^{(n)})K_n(\eta_0^{(n)},\eta_{2i+1}^{(n)},x,y)+P(\eta_{2i+1}^{(n)},\eta_0^{(n)})K_n(\eta_{2i+1}^{(n)},\eta_0^{(n)},x,y))\\&+\frac{1}{n^2}\sum_{i=0}^{m-1}(P(\eta_n^{(n)},\eta_{2i+1}^{(n)})K_n(\eta_n^{(n)},\eta_{2i+1}^{(n)},x,y)+P(\eta_{2i+1}^{(n)},\eta_n^{(n)})K_n(\eta_{2i+1}^{(n)},\eta_n^{(n)},x,y))\\&+\frac{2}{n^2}\sum_{j=1}^{m-1}\sum_{i=0}^{m-1}(P(\eta_{2i+1}^{(n)},\eta_{2j}^{(n)})K_n(\eta_{2i+1}^{(n)},\eta_{2j}^{(n)},x,y)+P(\eta_{2j}^{(n)},\eta_{2i+1}^{(n)})K_n(\eta_{2j}^{(n)},\eta_{2i+1}^{(n)},x,y)),\end{aligned}\tag{2.6}$$

when $n=2m$ and

$$\begin{aligned}P(x,y)=&\frac{1}{2n^2}(P(\eta_0^{(n)},\eta_0^{(n)})K_n(\eta_0^{(n)},\eta_0^{(n)},x,y)+P(\eta_n^{(n)},\eta_n^{(n)})K_n(\eta_n^{(n)},\eta_n^{(n)},x,y))\\&+\frac{1}{n^2}\sum_{i=1}^{m-1}(P(\eta_0^{(n)},\eta_{2i}^{(n)})K_n(\eta_0^{(n)},\eta_{2i}^{(n)},x,y)+P(\eta_{2i-1}^{(n)},\eta_n^{(n)})K_n(\eta_{2i-1}^{(n)},\eta_n^{(n)},x,y))\\&+\frac{1}{n^2}\sum_{i=1}^{m-1}(P(\eta_n^{(n)},\eta_{2i-1}^{(n)})K_n(\eta_n^{(n)},\eta_{2i-1}^{(n)},x,y)+P(\eta_{2i}^{(n)},\eta_0^{(n)})K_n(\eta_{2i}^{(n)},\eta_0^{(n)},x,y))\\&+\frac{2}{n^2}\sum_{j=1}^{m-1}\sum_{i=1}^{m-1}(P(\eta_{2i}^{(n)},\eta_{2j}^{(n)})K_n(\eta_{2i}^{(n)},\eta_{2j}^{(n)},x,y)+P(\eta_{2i-1}^{(n)},\eta_{2j-1}^{(n)})K_n(\eta_{2i-1}^{(n)},\eta_{2j-1}^{(n)},x,y)),\end{aligned}$$

when $n=2m-1$.

Here

$$\eta_j^{(n)}:=\cos\frac{j\pi}{n},\quad j=0,\dots,n,$$

$$\begin{aligned}K_n(x_1,y_1,x,y)=&D_n(\theta_1+\varphi_1,\theta_2+\varphi_2)+D_n(\theta_1+\varphi_1,\theta_2-\varphi_2)\\&+D_n(\theta_1-\varphi_1,\theta_2+\varphi_2)+D_n(\theta_1-\varphi_1,\theta_2-\varphi_2),\end{aligned}\tag{2.7}$$

$$(x_1,y_1):=(\cos\theta_1,\cos\theta_2),$$

$$(x,y):=(\cos\varphi_1,\cos\varphi_2),$$

and

$$D_n(\theta_1,\theta_2)=\frac{1}{2}\frac{\cos(n-1/2)\theta_1\cos(\theta_1/2)-\cos(n-1/2)\theta_2\cos(\theta_2/2)}{\cos\theta_1-\cos\theta_2}.$$

We could give also the multivariate interpolation formulae corresponding to the obtained before multivariate cubatures.

3. Proof of the results

Proof of Theorem 2.1. A repeat application of (2.1) with respect consecutively to x and y yields the cubature formula

$$I(f)\approx L_xL_y(f)=(E_x+\mathrm{O}_x)(E_y+\mathrm{O}_y)(f)=A(f)+B(f)$$

with an algebraic degree of precision $2n-1$ (i.e. it is exact for all $f\in\pi_{2n-1}(\mathbb{R}^2)$). Clearly, our theorem will be proved if we show that

$$A(f)=B(f)\quad \text{for all } f\in\pi_{2n-1}(\mathbb{R}^2). \tag{3.1}$$

Since the polynomials $\{P_{l-k}(x)P_k(y)\}_{l=0,k=0}^{2n-1\ \ l}$ constitute an orthogonal basis in $\pi_{2n-1}(\mathbb{R}^2)$ on $[-1,1]^2$, it is sufficient to verify the equality (3.1) only for $P_{l-k}(x)P_k(y)$, $l=0,\dots,2n-1$, $k=0,\dots,l$. We may assume without loss of generality that $P_0(x)=1$. Consider separately the following three situations:

(i) Let $f=1$. Then

$$\begin{aligned}A(f)&=\mathrm{O}_x\Big(\sum_{\mathrm{o}}a_j\Big)+E_x\Big(\sum_{\mathrm{e}}a_j\Big)\\&=\Big(\sum_{\mathrm{o}}a_j\Big)\Big(\sum_{\mathrm{o}}a_j\Big)+\Big(\sum_{\mathrm{e}}a_j\Big)\Big(\sum_{\mathrm{e}}a_j\Big)=2\Big(\sum_{\mathrm{o}}a_j\Big)^2,\end{aligned}$$

where we have used (2.2) in the last equality. Similarly, one obtains

$$\begin{aligned}B(f)&=\mathrm{O}_x\Big(\sum_{\mathrm{e}}a_j\Big)+E_x\Big(\sum_{\mathrm{o}}a_j\Big)\\&=2\Big(\sum_{\mathrm{o}}a_j\Big)\Big(\sum_{\mathrm{e}}a_j\Big)=2\Big(\sum_{\mathrm{o}}a_j\Big)^2=A(f)\end{aligned}$$

and (3.1) holds in this case.

(ii) Let $f(x,y)=P_k(y)$ and $k>0$. Then

$$\begin{aligned}A(f)&=\mathrm{O}_x\mathrm{O}_y(P_k)+E_xE_y(P_k)\\&=\Big(\sum_{\mathrm{o}}a_j\Big)\mathrm{O}_y(P_k)+\Big(\sum_{\mathrm{e}}a_j\Big)E_y(P_k)=\Big(\sum_{\mathrm{o}}a_j\Big)L(P_k),\end{aligned}$$

where we have used (2.2) only. In exactly the same way we find an expression for $B(f)$ which coincides with that for $A(f)$.

The case $f(x,y)=P_k(x)$ and $k>0$ goes similarly.

(iii) Let $f(x,y)=P_{l-k}(x)P_k(y)$ with $0<k<l\leqslant 2n-1$. Then

$$A(f)=\mathrm{O}_x(P_{l-k})\mathrm{O}_y(P_k)+E_x(P_{l-k})E_y(P_k),$$
$$B(f)=\mathrm{O}_x(P_{l-k})E_y(P_k)+E_x(P_{l-k})\mathrm{O}_y(P_k).$$

Since $I'(P_k)=0$ for $k>0$ (because of the orthogonality), we have

$$0=I'(P_k)=L(P_k)=E(P_k)+\mathrm{O}(P_k) \quad \text{for } k=1,\dots,2n-1,$$

which in view of (2.3) implies

$$\mathrm{O}(P_k)=-E(P_k)=0 \quad \text{for } k=1,\dots,n-1,$$

and then, clearly, $A(f)=B(f)=0$ for $k\leqslant n-1$.

If $k>n-1$, then $0<l-k\leqslant n-1$ and hence, $\mathrm{O}(P_{l-k})=-E(P_{l-k})=0$, which also yields $A(f)=B(f)=0$. The proof is complete. □

Proof of Corollary 2.2. Consider the Gauss–Lobatto quadrature formula

$$\int_{-1}^{1} w_0(x)g(x)\,\mathrm{d}x \approx \frac{\pi}{n}\left(\frac{1}{2}g(\eta_0^{(n)})+\sum_{j=1}^{n-1} g(\eta_j^{(n)})+\frac{1}{2}g(\eta_n^{(n)})\right).$$

Recall that $\eta_j^{(n)}:=\cos j\pi/n$, $j=0,\dots,n$ are the extremal points of the Tchebycheff polynomial of first kind $T_n(x)$, namely,

$$T_n(x)=\cos(n\arccos x), \quad x\in[-1,1].$$

We need only to check the conditions in Theorem 2.1. Clearly, (2.2) holds for every n. Let us prove (2.3).

Assume $n=2m$. Note that

$$\eta_{2j-1}^{(2m)}=\cos\frac{(2j-1)\pi}{2m}=\xi_j^{(m)}, \quad j=1,\dots,m,$$

where $\xi_j^{(m)}$ are the zeros of $T_m(x)$. Since the classical Mehler–Hermite quadrature formula

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \approx \frac{\pi}{m}\sum_{j=1}^{m} f(\xi_j^{(m)})$$

is exact for $f\in\pi_{2m-1}$, we have (because of the orthogonality)

$$0=\sum_{j=1}^{m} T_k(\xi_j^{(m)}), \quad k=1,\dots,n-1.$$

Therefore,

$$\mathrm{O}(T_k)=\sum_{j=1}^{m} T_k(\eta_{2j-1}^{(n)})=\sum_{j=1}^{m} T_k(\xi_j^{(m)})=0, \quad k=1,\dots,n-1.$$

This, combined with the fact that $L(T_k)=0,\quad k=1,\dots,2n-1$, gives

$$E(T_k)=0,\quad k=1,\dots,n-1.$$

Assume now that $n=2m-1$. Then

$$E(T_k)=\frac{\pi}{n}\left(\frac{1}{2}+\sum_{j=1}^{m-1}T_k(\eta_{2j}^{(2m-1)})\right)=\frac{\pi}{n}\left(\frac{1}{2}+\sum_{j=1}^{m-1}\cos j\frac{2k\pi}{2m-1}\right)=0.$$

The corresponding formula (2.4) is minimal when $n=2m$ and near minimal when $n=2m-1$ as follows from Möller's estimate ($N^*(2m)\geqslant 2m^2+2m$ and $N^*(2m-1)\geqslant 2m^2-1$). (2.5) is near minimal in all cases, since it uses $2m^2+2m+1$ nodes when $n=2m$ and $2m^2$ nodes when $n=2m-1$. □

Proof of Theorem 2.3. Let $d=2v$. The cubature formula

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\approx L_1\cdots L_d(f) \tag{3.2}$$

is exact for all polynomials of degree $2n-1$ in d variables. But (3.2) can be rewritten as

$$\begin{aligned}&\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\quad=\int_{B^{d-2}} W_0^{(d-2)}(\boldsymbol{x}_{d-2})\int\int_{[-1,1]^2} w_0(x)w_0(y)f(\boldsymbol{x}_{d-2},x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\boldsymbol{x}_{d-2}\\ &\quad\approx L_1\cdots L_{n-2}[L_xL_yf(\boldsymbol{x}_{d-2},x,y)],\end{aligned}$$

and by the Corollary 2.2

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\approx 2^v A_{12}A_{34}\cdots A_{d-1,d}(f).$$

Similarly when $d=2v+1$ we get

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\approx 2^v A_{12}A_{34}\cdots A_{d-2,d-1}L(f),$$

or, more generally,

$$\int_{B^d} W_0^{(d)}(\boldsymbol{x})f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\approx 2^v A_{12}\cdots A_{2s-1,2s}L_{2s+1}A_{2s+2,2s+3}\cdots A_{d-1,d}(f).$$

The proof is completed. □

Proof of Theorem 2.4. We are going to apply an analogue of the following fact, concerning univariate polynomials.

Assume that $\{p_k\}_{k=0}^{\infty}$, $p_k=\alpha_k x^k+\cdots$, is a normalized orthogonal system of polynomials on $[a,b]$ with respect to the weight $\mu(x)$. Let

$$\int_a^b \mu(t)f(t)\,\mathrm{d}t\approx\sum_{j=1}^{n}a_jf(x_j)$$

be the quadrature formula of algebraic degree of precision $2n-1$. Then for every $f\in\pi_{n-1}(\mathbb{R})$ we have

$$f(x)=\sum_{k=0}^{n-1}\left[\int_a^b \mu(t)f(t)p_k(t)\,\mathrm{d}t\right]p_k(x)$$
$$=\sum_{k=0}^{n-1}\sum_{j=1}^{n}a_j f(x_j)p_k(x_j)p_k(x)$$
$$=\sum_{j=1}^{n}f(x_j)a_j\sum_{k=0}^{n-1}p_k(x_j)p_k(x).$$

Since the dimension of $\pi_{n-1}(\mathbb{R})$ equals n and every $f\in\pi_{n-1}(\mathbb{R})$ was presented as a linear combination of the polynomials

$$l_{nj}(x):=a_j\sum_{k=0}^{n-1}p_k(x_j)p_k(x),\quad j=1,\dots,n,$$

we conclude that $\{l_{nj}\}_{j=1}^{n}$ are the fundamental polynomials for the Lagrange interpolation with nodes $x_1,\dots,x_n$. Note that by the Christoffel–Darboux formula one can get

$$l_{nj}(x)=a_j p_{n-1}(x_j)\frac{\alpha_{n-1}}{\alpha_n}\frac{p_n(x)}{(x-x_j)}$$

which leads to the usual presentation

$$l_{nj}(x)=\frac{p_n(x)}{(x-x_j)p'(x_j)}$$

of l_{nj}. We can apply the above-mentioned reasoning to get explicit expressions for the fundamental polynomials for the Lagrange-type interpolation. In addition, if we have a certain analogue of the Christoffel–Darboux formula, we may hope to get a compact form of these expressions. Next, we sketch the calculations and present the final results in this particular case.

Every $P(x,y)\in\pi_{n-1}(\mathbb{R}^2)$ can be written as

$$P(x,y)=\sum_{l=0}^{n-1}\sum_{k=0}^{l}a_{lk}(P)T_{l-k}(x)T_k(y), \tag{3.3}$$

where

$$a_{lk}(P)=\frac{1}{\gamma_{lk}}\iint_{[-1,1]^2}W_0^{(2)}(t,s)P(t,s)T_{l-k}(t)T_k(s)\,\mathrm{d}t\,\mathrm{d}s$$

and

$$\gamma_{lk}=\iint_{[-1,1]^2}W_0^{(2)}(t,s)T_{l-k}^2(t)T_k^2(s)\,\mathrm{d}t\,\mathrm{d}s.$$

Denote

$$K_n(x_1, y_1, x, y) := \pi^2 \sum_{l=0}^{n-1} \sum_{k=0}^{l} \frac{1}{\gamma_{lk}} T_{l-k}(x_1) T_k(y_1) T_{l-k}(x) T_k(y).$$

This is the Christoffel–Darboux sum, corresponding to our case. Note that Xu found in [3] the explicit expression (2.7) for this sum, which provides actually an extension of the classical Christoffel–Darboux formula.

Next, we apply the already found cubatures (for even and odd n) to compute the Fourier–Tchebycheff coefficients $a_{lk}(P)$ of P. Therefore, if we return back in (3.3), change the summation and use the form of K_n, we obtain the interpolation formulae (2.6). This way, using different tools we get in a simpler way the results from [4]. □

4. Concluding remarks

(1) The cubature (2.4) for $n = 2m$ is established by Morrow and Patterson [2] and Xu [4]. For the first time the explicit form of (2.5) in the case $n = 2m - 1$ is given by Xu in [4]. The two other formulae ((2.4) for $n = 2m - 1$ and (2.5) for $n = 2m$) have not been mentioned in [4].

(2) In each of the formulae, derived in Theorem 2.3 we can use B_{pq} instead of A_{pq} to derive new examples of cubatures.

References

[1] H.M. Möller, Kubaturformeln mit minimaler Knotenzahl, Numer. Math. 25 (1976) 185–200.

[2] C.R. Morrow, T.N.L. Patterson, Construction of algebraic cubature rules using polynomial ideal theory, SIAM J. Numer. Anal. 15 (1978) 953–976.

[3] Y. Xu, Christoffel functions and Fourier series for multivariate orthogonal polynomials, J. Approx. Theory 82 (1995) 205–239.

[4] Y. Xu, Lagrange interpolation on Chebyshev points of two variables, J. Approx. Theory 87 (1996) 220–238.