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Numerical integration over a disc. A new Gaussian quadrature formula

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Summary. We construct a quadrature formula for integration on the unit disc which is based on line integrals over n distinct chords in the disc and integrates exactly all polynomials in two variables of total degree $2n-1$.

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1. Introduction

The ordinary type of information data for approximation of functions f or functionals of them in the univariate case consists of function values $\{f(x_1), \ldots, f(x_m)\}$. The classical Lagrange interpolation formula and the Gauss quadrature formula are famous examples. The simplicity of the approximation rules, their universality, the elegancy of the proofs and the beauty of these classical results show that the function values are really the most natural pieces of information in the reconstruction of functions and functionals. The direct transformation of the univariate results to the multivariate setting faces however various difficulties. For example, the problem of constructing a polynomial $P(x, y)$ of degree n,

$$P(x,y) = \sum_{i+j\le n} a_{ij} x^i y^j$$

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satisfying given Lagrangean interpolation conditions

$$P(x_j, y_j) = f_j, \quad j = 1, \dots, \binom{n+2}{2},$$

leads to a linear system in unknowns $\{a_{ij}\}$ which has not always a unique solution. A more convincing example is the long years struggle of many mathematicians to construct cubature formulas of the form

$$\int\!\!\int_{\Omega} f(x,y)\,dx\,dy \approx \sum_{j=1}^{N} C_j f(x_j, y_j)$$

with preassigned number N of nodes that integrate exactly all polynomials $P(x,y)$ of degree as high as possible over a given simple domain Ω. Only a few cubatures of this type are known explicitly. This indicates that the setting of the multivariate approximation problems did not reach yet its most natural formulation and in particular, the assertion that the sampling of function values is the most natural basis for recovery of functions should be met with a certain doubt. The research practice in mathematics shows that the ”naturally” posed problems have nice solutions and far going extensions. What is then the most natural type of information for reconstruction of functions in $\mathbb{R}^2$. The recent development in tomography, as well as the power of the Radon transform and other results in multivariate interpolation suggest as a reasonable choice the data of mean values

$$\{\int_{I_k} f\},$$

where $\{I_k\}$ are line segments. A remarkable result in this direction is the Hakopian interpolation formula (see [1]) which can be viewed as a multivariate extension of the Lagrange interpolation formula. It seems that many classical approximation problems in the univariate case dealing with point evaluations should admit natural extensions in the multivariate case (i.e., in $\mathbb{R}^d$) if the approximation is based on integrals over hyperplanes of dimension $d-1$.

In this paper we consider the extremal problem of Gauss about quadrature formulas of highest algebraic degree of precision, formulated in an appropriate form for numerical integration over the unit disc $D := \{(x,y) : x^2 + y^2 \le 1\}$ with respect to a mean value information data. Precisely, we construct a quadrature formula of the form

$$\int\!\!\int_{D} f(x,y)\,dx\,dy \approx \sum_{k=1}^{n} \lambda_k \int_{I_k} f$$

of highest degree of precision with respect to the class of algebraic polynomials of two variables.

2. Preliminaries

As usual, we use the notation

$$\pi_n(\mathbb{R}^2) := \Big\{ \sum_{i+j\le n} a_{ij} x^i y^j \Big\}$$

for the set of all algebraic polynomials of total degree n. The dimension of $\pi_n(\mathbb{R}^2)$ is equal to $\binom{n+2}{2}$.

Let Ω be a given bounded domain in the plane $\mathbb{R}^2$. Our main result concerns the case when Ω is the unit disc D. We shall consider integrable functions $f(x,y)$ on Ω. For the sake of simplicity, we suppose that $f(x,y)$ is supported on Ω, that is, $f(x,y)$ vanishes outside Ω. Any pair of parameters (t,θ) defines a line

$$I(t,\theta) := \{(x,y) : x\cos\theta + y\sin\theta = t\}.$$

We assume that $\theta \in [0,\pi)$ and

$$I(t,\theta) \cap \Omega \not\equiv \emptyset.$$

The *projection* $P_f(t,\theta)$ of f along the line $I(t,\theta)$ is defined by

$$P_f(t,\theta) := \int_{-\infty}^{+\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

Sometimes, instead of $P_f(t,\theta)$, we shall use the notation

$$\int_{I(t,\theta)} f(x,y)\, ds$$

for the projection when we want to stress that the integral is taken over the line segment $I(t,\theta) \cap \Omega$.

Given the parameters (t_k,θ_k), we set

$$I_k := I(t_k,\theta_k), \quad k = 1,\dots,n,$$

$$I := \{I_1,\dots,I_n\},$$

$$L_{I_k} \equiv L_{t_k,\theta_k}(x,y) := x\cos\theta_k + y\sin\theta_k - t_k.$$

Clearly $L_{I_k} \in \pi_1(\mathbb{R}^2)$.

Assume that I_k are n distinct lines defined by the parameters $\{(t_k,\theta_k)\}_{k=1}^n$. We shall study quadrature formulas of the form

(1) $$\iint_\Omega f(x,y)\, dx\, dy \approx \sum_{k=1}^n A_k \int_{I_k} f(x,y)\, ds$$

in the class of integrable functions $L^1(\Omega)$. Here A_k are real coefficients. The *algebraic degree of precision* of the quadrature (1) (abbreviated to ADP(1) is the maximal integer m so that the quadrature (1) integrates exactly all polynomials $P(x,y)$ of degree less than or equal to m. Our first observation is:

$$\text{ADP}(1) < 2n \tag{2}$$

for each choice of the coefficients A_k and the parameters (t_k, θ_k). To show this, we introduce the polynomial

$$\omega(x,y) = \omega(I;x,y) := \prod_{k=1}^{n} L_{I_k}(x,y).$$

Clearly $\omega^2(x,y) \in \pi_{2n}(\mathbb{R}^2)$ and $\omega^2(x,y) \geq 0$. Thus

$$\iint_{\Omega} \omega^2(x,y)\,dx\,dy > 0, \quad \text{while} \quad \sum_{k=1}^{n} A_k \int_{I_k} \omega^2(x,y)\,ds = 0.$$

Hence the quadrature (1) is not exact for $\omega^2(x,y)$ and this proves (2).

The maximal ADP that could be achieved by a quadrature (1), using n evaluations of the projection is $2n-1$. Does there exist a quadrature of ADP equal to $2n-1$? There is no strong evidence for an affirmative answer, since the dimension of the space of polynomials of degree $2n-1$ is $\binom{2n+1}{2}$ and hence much greater than the number $3n$ of the parameters, used in the quadrature (A_k, t_k, θ_k, $k = 1,\ldots,n$). However, we construct here a quadrature with ADP $= 2n-1$ in the case $\Omega = D$.

Next we derive a simple necessary condition for (1) to have ADP $=$ $2n-1$.

Lemma 1. *If the quadrature formula (1) is exact for all polynomials of degree $2n-1$ then the polynomial $\omega(x,y) = L_{I_1}(x,y)\cdots L_{I_n}(x,y)$ is orthogonal to any $Q \in \pi_{n-1}(\mathbb{R}^2)$ in Ω.*

Proof. For an arbitrary element $Q \in \pi_{n-1}(\mathbb{R}^2)$

$$\omega(x,y)Q(x,y) \in \pi_{2n-1}(\mathbb{R}^2)$$

and therefore (1) will be exact for it. Namely,

$$\begin{aligned}\iint_{\Omega} \omega(x,y)Q(x,y)\,dx\,dy &= \sum_{k=1}^{n} A_k \int_{I_k} \prod_{j=1}^{n} L_{I_j}(x,y)Q(x,y)\,ds \\ &= \sum_{k=1}^{n} A_k.0 = 0,\end{aligned}$$

which proves the lemma. □

A quadrature formula of the form (1) with ADP $= 2n-1$ will be called *Gaussian*. Lemma 1 asserts that we can construct such a formula only for domains Ω, for which one can find a polynomial of the form

$$\prod_{k=1}^{n}(a_k x + b_k y + c_k), \qquad \text{with } a_k,\ b_k,\ c_k \in \mathbb{R},$$

which is orthogonal on Ω to all polynomials from $\pi_{n-1}(\mathbb{R}^2)$.

Lemma 2. *If (1) is a Gaussian quadrature formula then $A_k > 0$ for $k = 1,\dots,n$.*

Proof. Construct the polynomials

$$\omega_k(x,y) = \omega_k(I;x,y) := \prod_{j=1, j\neq k}^{n} L_{I_j}(x,y), \quad k = 1,\dots,n.$$

Since (1) is Gaussian and $\omega_k^2(x,y) \in \pi_{2n-2}(\mathbb{R}^2)$, we have

$$\iint_\Omega \omega_k^2(x,y)\,dx\,dy = \sum_{j=1}^{n} A_j \int_{I_j} \omega_k^2(x,y)\,ds = A_k \int_{I_k} \omega_k^2(x,y)\,ds,$$

which implies that $A_k > 0$ for $k = 1,\dots,n$. $\square$

The following lemma gives a relation between two Gaussian quadrature formulas.

Lemma 3. *Assume that (1) and the quadrature formula*

$$\iint_\Omega f(x,y)\,dx\,dy \approx \sum_{k=1}^{n} B_k \int_{J_k} f(x,y)\,ds$$

are Gaussian. Then

$$J_k \cap \Big(\big(\bigcup_{l=1, l\neq k}^{n} J_l\big) \cup \big(\bigcup_{l=1}^{n} I_l\big) \Big) \cap \Omega \not\equiv \emptyset, \quad k = 1,\dots,n.$$

Proof. Consider the polynomial

$$\omega_k(J;x,y) = \prod_{l=1, l\neq k}^{n} L_{J_l}(x,y), \quad k = 1,\dots,n.$$

Apply now the both quadratures to

$$\omega(I;x,y)\omega_k(J;x,y) \in \pi_{2n-1}(\mathbb{R}^2), \quad k = 1,\dots,n.$$

We have

$$\begin{aligned} 0 &= \iint_{\Omega} \omega(I; x, y)\omega_k(J; x, y)\, dx\, dy \\ &= \sum_{l=1}^{n} B_l \int_{J_l} \omega(I; x, y)\omega_k(J; x, y)\, ds \\ &= B_k \int_{J_k} \omega(I; x, y)\omega_k(J; x, y)\, ds. \end{aligned}$$

But according to the mean value theorem

$$\int_{J_k} \omega(I; x, y)\omega_k(J; x, y)\, ds = \omega(I; M_k)\omega_k(J; M_k) d(J_k \cap \Omega),$$

where $M_k \in J_k$ and $d(J_k \cap \Omega)$ is the length of the line segment $J_k \cap \Omega$. Finally,

$$M_k \in \Big(\big(\bigcup_{l=1, l\neq k}^{n} J_l\big) \cup \big(\bigcup_{l=1}^{n} I_l\big)\Big) \cap \Omega$$

and the proof is completed. □

3. Gaussian quadrature

Denote by $U_n(t)$ the Tchebycheff polynomial of second kind of degree n, i.e.,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where} \quad x = \cos\theta.$$

Let $\eta_1, \ldots, \eta_n$ be the zeroes of $U_n(x)$, that is, $\eta_k = \cos\frac{k\pi}{n+1}$, $k = 1, \ldots, n$.

Theorem 1. *The quadrature formula*

$$(*) \qquad \iint_D f(x, y)\, dx\, dy \approx \sum_{k=1}^{n} A_k \int_{-\sqrt{1-\eta_k^2}}^{\sqrt{1-\eta_k^2}} f(\eta_k, y)\, dy,$$

with

$$A_k = \frac{\pi}{n+1} \sin\frac{k\pi}{n+1} \qquad k = 1, \ldots, n,$$

is exact for each polynomial $f \in \pi_{2n-1}(\mathbb{R}^2)$.

Proof. Assume that $f(x, y)$ is any polynomial from $\pi_{2n-1}(\mathbb{R}^2)$. Then it can be presented in the form

$$f(x, y) = \sum_{k=0}^{2n-1} a_k(x) y^k, \quad \text{where} \ a_k(x) \in \pi_{2n-k-1}(\mathbb{R})$$

and thus

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sum_{k=0}^{2n-1} a_k(x) y^k\,dy\,dx$$
$$= \sum_{k=0,k=2s}^{2n-1} \frac{2}{2s+1}\int_{-1}^{1}\sqrt{1-x^2}a_k(x)(1-x^2)^s\,dx.$$

Observe that the univariate polynomials $a_k(x)(1-x^2)^s$ under the integral sign are of degree less than or equal to $2n-1$. Then we can compute the integral exactly using the classical Gaussian quadrature formula on $[-1,\,1]$ with weight $\xi(t)=\sqrt{1-t^2}$. Precisely,

$$\int_{-1}^{1}\sqrt{1-x^2}a_k(x)(1-x^2)^s\,dx = \sum_{l=1}^{n}\frac{\pi}{n}a_k(\eta_l)(1-\eta_l^2)^s.$$

Therefore

$$\iint_D f(x,y)\,dx\,dy = \sum_{l=1}^{n}\frac{\pi}{n+1}\sin^2\frac{l\pi}{n+1}\sum_{k=0,k=2s}^{2n-1}\frac{2}{2s+1}(1-\eta_l^2)^s a_k(\eta_l)$$
$$= \sum_{l=1}^{n}\frac{\pi}{n+1}\sin\frac{l\pi}{n+1}\sum_{k=0,k=2s}^{2n-1}a_k(\eta_l)\frac{2}{2s+1}\left(\sqrt{1-\eta_l^2}\,\right)^{2s+1}$$
$$= \sum_{l=1}^{n}\frac{\pi}{n+1}\sin\frac{l\pi}{n+1}\int_{-\sqrt{1-\eta_l^2}}^{\sqrt{1-\eta_l^2}}\sum_{k=0}^{2n-1}a_k(\eta_l)y^k\,dy$$
$$= \sum_{l=1}^{n}\frac{\pi}{n+1}\sin\frac{l\pi}{n+1}\int_{-\sqrt{1-\eta_l^2}}^{\sqrt{1-\eta_l^2}} f(\eta_l,y)\,dy$$

and the proof is completed. □

4. Ridge polynomials. Orthogonal polynomials

Given the real function $\rho(t)$ on $I\!R$ and a parameter $\theta \in [0,\pi)$ we define on $I\!R^2$ the associated *ridge function* $\rho(\theta;x,y)$ with direction θ in the following way

$$\rho(\theta;x,y) := \rho(x\cos\theta + y\sin\theta).$$

Clearly the ridge function is constant along any line of direction θ. As a consequence of this, one can compute

$$\iint_D \rho(\theta;x,y)\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}}\rho(t)\,ds\,dt = 2\int_{-1}^{1}\sqrt{1-t^2}\rho(t)\,dt$$

and thus, by the Gaussian quadrature,

$$\int\int_D \rho_{2n-1}(\theta; x, y)\, dx\, dy = \frac{2\pi}{n+1} \sum_{k=1}^{n} \sin^2 \frac{k\pi}{n+1} \rho_{2n-1}(\eta_k)$$

for every ridge polynomial $\rho_{2n-1}(\theta; x, y)$ of degree $2n-1$. This conclusion is just a particular case of Theorem 1, applied to the ridge polynomial ρ_{2n-1}. It shows that the use of a certain ridge polynimial basis in $\pi_N(\mathbb{R}^2)$ could produce a simple integration rule. The following is a well-known fact (see for example [3]).

Lemma 4. *Every polynomial of degree n in x and y is a sum of $n+1$ ridge polynomials (with any preassigned directions $\theta_0, \theta_1, \ldots, \theta_n$ which are distinct modulo π) of degree n.*

The ridge polynomials, associated with the Tchebycheff polynomial $U_n(t)$ play an essential role in this study. A simple consequence of Theorem 1 and Lemma 1 is the following

Corollary 1.

$$\int\int_D U_n(\theta; x, y)\, P(x, y)\, dx\, dy = 0 \tag{3}$$

for each $P(x, y) \in \pi_{n-1}(\mathbb{R}^2)$ and θ.

Now we give an explicit formula for the projection of a ridge polynomial.

Lemma 5. *Let $Q(t)$ be a polynomial of degree n presented in the form*

$$Q(t) = \sum_{j=0}^{n} a_j U_j(t).$$

Then the projection of the associate ridge polynomial $Q(\alpha; x, y)$ of direction α along the line $I(t, \theta)$ is given by the formula

$$P_Q(t, \theta) = 2\sqrt{1-t^2} \sum_{j=0}^{n} \frac{a_j}{j+1} U_j(t) \frac{\sin(j+1)(\theta-\alpha)}{\sin(\theta-\alpha)}.$$

Proof. It suffices to proof the lemma for $\alpha = 0$. So, we assume further that $\alpha = 0$.

Clearly

$$P_Q(t, \theta) = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} Q(t\cos\theta - y\sin\theta)\, dy$$

and after the substitution $t = \cos\tau$ we get

$$P_Q(\cos\tau, \theta) = \int_{-\sin\tau}^{\sin\tau} Q(\cos\tau\cos\theta - y\sin\theta)\, dy.$$

Setting $u = \cos\tau\cos\theta - y\sin\theta$, we arrive at the formula

$$P_Q(\cos\tau, \theta) = \frac{1}{\sin\theta}\int_{\cos(\tau+\theta)}^{\cos(\tau-\theta)} Q(u)\, du,$$

which can be found also in [3]. Now we perform the integration, taking into account the given presentation of $Q(t)$ and the relation $T'_{j+1}(t) = (j+1)U_j(t)$ between the Tchebycheff polynomials of first and second kind. We have

$$\begin{aligned} P_Q(\cos\tau, \theta) &= \frac{1}{\sin\theta}\sum_{j=0}^{n}\frac{a_j}{j+1}\left\{T_{j+1}(\cos(\tau-\theta)) - T_{j+1}(\cos(\tau+\theta))\right\} \\ &= 2\sum_{j=0}^{n}\frac{a_j}{j+1}\frac{\sin(j+1)\tau\ \sin(j+1)\theta}{\sin\theta} \\ &= 2\sum_{j=0}^{n}\frac{a_j}{j+1}\ \frac{\sin(j+1)\tau}{\sin\tau}\ \frac{\sin(j+1)\theta}{\sin\theta}\ \sin\tau \\ &= 2\sum_{j=0}^{n}\frac{a_j}{j+1}U_j(t)\sqrt{1-t^2}\ \frac{\sin(j+1)\theta}{\sin\theta}, \end{aligned}$$

which is the wanted formula. The proof is completed. □

An immediate consequence from Lemma 5 is the following integration formula, derived in [2].

Corollary 2. *For each t and θ,*

$$\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} U_m(t\cos\theta + s\sin\theta)\, ds = \frac{2}{m+1}\sqrt{1-t^2}\, U_m(t)\ \frac{\sin(m+1)\theta}{\sin\theta}.$$

A similar formula can be given for the projection of any polynomial $f \in \pi_n(\mathbb{R}^2)$. Indeed, according to Lemma 4, f can be written in the form

$$f(x,y) = \sum_{i=0}^{n} b_i Q_i(\theta_i; x, y), \tag{4}$$

and consequently,

$$f(x,y) = \sum_{i=0}^{n}\sum_{m=0}^{n} b_i a_{im} U_m(\theta_i; x, y),$$

where

$$Q_i(t) = \sum_{m=0}^{n} a_{im} U_m(t). \tag{5}$$

Then, by Corollary 2,

$$\begin{aligned} P_f(t,\theta) &= \sum_{i=0}^{n} \left\{ \sum_{m=0}^{n} \frac{b_i a_{im}}{m+1} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} U_m(t\cos\theta + s\sin\theta)\, ds \right\} \\ &= 2\sqrt{1-t^2} \sum_{m=0}^{n} \sum_{i=0}^{n} c_{im} \frac{\sin(m+1)(\theta-\theta_i)}{\sin(\theta-\theta_i)} U_m(t) \end{aligned}$$

with $c_{im} = b_i a_{im}/(m+1)$. Thus, we proved the following

Corollary 3. *Let f be any polynomial from $\pi_n(\mathbb{R}^2)$. Assume that f is given by (4) with Q_i presented in the form (5). Then*

$$P_f(t,\theta) = 2\sqrt{1-t^2} \sum_{m=0}^{n} \sum_{i=0}^{n} \frac{b_i a_{im}}{m+1} \frac{\sin(m+1)(\theta-\theta_i)}{\sin(\theta-\theta_i)} U_m(t). \tag{6}$$

One can see from (6) that the projection of any polynomial $f \in \pi_n(\mathbb{R}^2)$ is a function of the form $\sqrt{1-t^2}\varphi(t)$ where $\varphi(t)$ is an algebraic polynomial of degree n. This fact was used already in the proof of Theorem 1. Moreover, (6) can be used to find the coefficients of this polynomial in terms of the directions $\{\theta_m\}$.

Using Corollary 2, one may construct an orthonormal basis of ridge polynomials in $\pi_n(\mathbb{R}^2)$, as shown in the next lemma.

Lemma 6. *Set $\theta_{mj} := \frac{j\pi}{m+1}$ for $j = 0,\dots,m$, $m = 0,\dots,n$. The ridge polynomials*

$$\frac{1}{\sqrt{\pi}} U_m(\theta_{mj}; x, y), \quad m = 0,\dots,n,\ j = 0,\dots,m\ , \tag{7}$$

form an orthonormal basis in $\pi_n(\mathbb{R}^2)$.

Proof. The relation

$$\iint_D U_m(\theta_{mj}; x, y)\, U_k(\theta_{ki}; x, y)\, dx\, dy = 0, \quad \text{for } m \neq k,$$

follows immediately from Corollary 1. In the case $m = k$, by Corollary 2,

$$\begin{aligned} &\iint_D U_m(\theta_{mj}; x, y)\, U_m(\theta_{mi}; x, y)\, dx\, dy \\ &= \int_{-1}^{1} U_m(t) \int_{I(t,\theta_{mj})} U_m(\theta_{mi}; x, y) ds \\ &= \frac{2}{m+1} \int_{-1}^{1} \sqrt{1-t^2}\, U_m^2(t)\, dy\, \frac{\sin(m+1)(\theta_{mi}-\theta_{mj})}{\sin(\theta_{mi}-\theta_{mj})} = \pi\delta_{ij}, \end{aligned}$$

since

$$\frac{\sin(m+1)(\theta_{mi}-\theta_{mj})}{\sin(\theta_{mi}-\theta_{mj})} = \frac{\sin(i-j)\pi}{\sin(i-j)\pi/(m+1)} = (m+1)\delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if i=j} \end{cases}.$$

The proof is completed. □

Lemma 1 shows that the orthogonal polynomials, as in the univariate case, are closely related with the construction of Gaussian quadratures.

We next discuss another orthonormal basis for D in $\pi_n(I\!R^2)$ (see [5]). These are the polynomials $F_{mj}(x,y), \quad m = 0,\dots,n,\ j = 0,\dots,m,$ defined via

(8)

$$F_{mj}(x,y) := Q_{n-k}(x,k)(1-x^2)^{\frac{k}{2}} P_k\left(\frac{y}{\sqrt{1-x^2}}\right) = Q_{n-k}(x,k)W_k(x,y),$$

where $P_k(t)$ is the $k-th$ Legendre polynomial and $Q_{n-k}(t,k)$ is the polynomial of degree $(n-k)$, orthogonal in $(-1,1)$ with weight $\xi_k(t) = (1-t^2)^{k+\frac{1}{2}}$. Since $Q_{n-k}(t,k)$ is a Jacobi polynomial with parameters $\alpha = \beta = k + \frac{1}{2}$ and for the Jaccobi polynomials $J_n^{(\alpha,\beta)}(t)$ the equality

$$\frac{d}{dt}\{J_n^{(\alpha,\beta)}(t)\} = \frac{1}{2}(n+\alpha+\beta+1)J_{n-1}^{(\alpha+1,\beta+1)}(t)$$

holds, then

$$Q_{n-k}(t,k) = c_k \frac{d^k}{dt^k} U_n(t), \quad k = 0,\dots,n, \quad c_k \in I\!R.$$

Note that

$$W_k(x,y) = \begin{cases} (y^2+x^2p_{2s,1}^2-p_{2s,1}^2)...(y^2+x^2p_{2s,s}^2-p_{2s,s}^2) \\ \qquad\qquad\qquad\qquad \text{if } k=2s \\ y(y^2+x^2p_{2s+1,1}^2-p_{2s+1,1}^2)...(y^2+x^2p_{2s+1,s}^2-p_{2s+1,s}^2) \\ \qquad\qquad\qquad\qquad \text{if } k=2s+1 \end{cases},$$

where $p_{k,l},\ l = 1,\dots,[\frac{k}{2}]$ are the positive zeroes of $P_k(t)$. Therefore the polynomials $F_{mj}(x,y)$ vanish on elipses. This fact leads to the conclusion, that the two orthogonal systems (7) and (8) are different.

The next result due to Marr [4] gives the projection of any orthogonal polynomial along any line $I(t,\theta)$.

Lemma 7. *Assume that $\omega(x,y)$ is a polynomial from $\pi_n(I\!R^2)$ which is orthogonal to each polynomial $Q \in \pi_{n-1}(I\!R^2)$ on the unit disc D. Then,*

$$\int_{I(t,\theta)} \omega(x,y)\,ds = \frac{2}{n+1}\sqrt{1-t^2}\,U_n(t)\omega(\cos\theta,\sin\theta).$$

The proof of this nice observation can be found in [4]. In particular, when $\theta = 0$

$$\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \omega(t,y)\,dy = A\sqrt{1-t^2}\,U_n(t) \quad t \in (-1,1), \quad A \in I\!R.$$

We can derive it here easily from our auxiliary results in the previous section. To do this, note that

$$\omega(x,y) = \sum_{j=0}^{n} b_j U_n(\theta_j; x, y)$$

with some coefficients $\{b_j\}$. Then, according to (6),

$$\begin{aligned}\int_{I(t,\theta)} \omega(x,y)\,ds &= \frac{2}{n+1}\sqrt{1-t^2}\,U_n(t)\sum_{j=0}^{n} b_j \frac{\sin(n+1)(\theta-\theta_{nj})}{\sin(\theta-\theta_{nj})} \\ &= \frac{2}{n+1}\sqrt{1-t^2}\,U_n(t)\omega(\cos\theta,\sin\theta).\end{aligned}$$

Theorem 2. *If the polynomial $\omega(x,y) := L_{I_1}(x,y)\cdots L_{I_n}(x,y)$ is orthogonal to $\pi_{n-1}(I\!R^2)$ then*

$$t_k \in \{\eta_1,\ldots,\eta_n\} \qquad \left(\eta_k = \cos\frac{k\pi}{n+1},\ k = 1,\ldots,n\right)$$

or $\cos(\theta_k - \theta_i) = t_i$ for some i.

Proof. From Marr's Lemma and the fact, that $\omega(x,y)$ vanishes on I_k, it follows that

$$\begin{aligned}0 = \int_{I_k} \omega(x,y)\,ds &= \frac{2}{n+1}\sqrt{1-t_k^2}\,U_n(t_k)\omega(\cos\theta_k,\sin\theta_k) \\ &= \frac{2}{n+1}\sqrt{1-t_k^2}\,U_n(t_k)\prod_{i=1}^{n}(\cos(\theta_k-\theta_i)-t_i).\end{aligned}$$

Since $t_k \in (-1,1)$, Theorem 2 is proved. $\square$

5. Characterization of the Gaussian quadratures

It seems that the quadrature formula given in Theorem 1 with ADP $= 2n-1$ is unique (up to rotation). In this section we give a certain characterization of the Gaussian quadratures which supports our suggestion. We derive a simple observation, that gives a necessary condition for the quadrature (1) for the unit disk D to have ADP $= 2n-1$. Let $A_1, \dots, A_s$ be the common points of the unit circle $\{(x,y) \ : \ x^2+y^2=1\}$ and the lines $I := \{I_1, \dots, I_n\}$, induced by (1), enumerated with respect to their natural position on the circle. Let $A_{s+1} \equiv A_1$. Then the folowing criterion is true.

Criterion. *If there are two consecutive points A_j and A_{j+1}, $j \in \{1, \dots, s\}$, such that the length of the arc, determined by them is greater than $\frac{2\pi}{n+1}$, then (1) is not Gaussian.*

Proof. Assume the opposite and consider the set Ψ of line segments $l_{\theta,j}$, $\theta \in [0, 2\pi)$, $1 \le j < \frac{n+1}{2}$, j - integer, with endpoints $(\cos\theta, \sin\theta)$ and $(\cos(\theta + \frac{2j\pi}{n+1}), \sin(\theta + \frac{2j\pi}{n+1}))$ and the set I. Then, there is a line segment $l_{\theta,j}$ with the property

$$l_{\theta,j} \cap I_k \equiv \emptyset \quad \text{for each } k = 1, \dots, n.$$

Without loss of generality $\theta = 0$ and therefore,

$$l_{0,j} \equiv I(\eta_j, 0).$$

But by Lemma 3, applied to the Gaussian formulas (1) and (*),

$$l_{0,j} \cap ((\bigcup_{k=1, k\neq j}^{n} I(\eta_k, 0)) \cup (\bigcup_{k=1}^{n} I_k)) \cap D \not\equiv \emptyset.$$

The proof is completed. $\square$

The following necessary condition is also true.

Lemma 8. *If (1) is a Gaussian quadrature formula for D, then the polynomial $\omega(x,y) := L_{I_1}(x,y) \dots L_{I_n}(x,y)$ has the property*

$$\omega(-x,-y) = (-1)^n \omega(x,y) \tag{9}$$

and when $n \ge 2$ there are indexes j, $k \in \{1, \dots, n\}$, such that

$$|\ t_j\ | \ge \frac{1}{2}, \quad \textit{and} \quad |\ t_k\ | \le \frac{1}{2}.$$

Proof. Lemma 1 shows that $\omega(x,y)$ is orthogonal to any $Q \in \pi_{n-1}(\mathbb{R}^2)$ in D with weight $p(x,y) \equiv 1$. The domain D and $p(x,y)$ are central symmetric and therefore (see [6]) $\omega(x,y)$ has the property (9).

When $n \geq 2$ ADP $= 2n-1 \geq 3$ and because (1) is exact for the polynomials $f_1(x,y) \equiv 1$, $f_2(x,y) = x^2$, $f_3(x,y) = y^2$, after a simple computation one can obtain

$$\sum_{k=1}^{n} A_k \sqrt{1-t_k^2} = \frac{\pi}{2},$$

$$\frac{2}{3}\sum_{k=1}^{n} A_k \sqrt{1-t_k^2}(3t_k^2\cos^2\theta_k + (1-t_k^2)\sin^2\theta_k) = \frac{\pi}{4},$$

$$\frac{2}{3}\sum_{k=1}^{n} A_k \sqrt{1-t_k^2}(3t_k^2\sin^2\theta_k + (1-t_k^2)\cos^2\theta_k) = \frac{\pi}{4}$$

and therefore,

$$\sum_{k=1}^{n} A_k \sqrt{1-t_k^2}(4t_k^2-1) = 0.$$

Since $t_k \in (-1,1)$ and, by Lemma 2, $A_k > 0,\ k = 1,\dots,n$, one has that there exist indexes $j,\ k \in \{1,\dots,n\}$, satisfying the above mentioned condition. $\square$

Actually Lemma 8 claims, that the set of lines $I := \{I_1,\dots,I_n\}$, induced by a Gaussian quadrature formula (1) is central symmetric with respect to the origin. Namely, after a suitable enumeration,

$$t_k = -t_{k+s}, \quad k = 1,\dots,s, \quad \text{when } n = 2s,$$

$$t_k = -t_{k+s+1}, \quad k = 1,\dots,s, \quad t_{s+1} = 0, \quad \text{when } n = 2s+1.$$

Taking into account all previous results a natural question comes up. Does the set

$$\Gamma_1 := \{f \in \pi_n(\mathbb{R}^2) : f = \prod_{k=1}^{n}(a_k x + b_k y + c_k),$$
$$f \text{ orthogonal to } \pi_{n-1}(\mathbb{R}^2)\}$$

coincide with the set

$$\Gamma_2 := \{f \in \pi_n(\mathbb{R}^2) : f = \prod_{k=1}^{n} L_{I_k}(x,y),$$
$$I_1,\dots,I_n \text{ are induced by a Gaussian quadrature (1)}\}.$$

Clearly, by Lemma 1, $\Gamma_2 \subseteq \Gamma_1$. There is an example (see [6]) of a polynomial $R(x,y) \in \Gamma_1$, such that its linear multipliers are lines, passing through the origin and the equidistant points $(\cos\frac{k\pi}{n}, \sin\frac{k\pi}{n}), k = 1,\dots,n$, on the unit circle $\{(x,y)\ :\ x^2+y^2 = 1\}$. Since each line generated by this polynomial passes through the origin, by the second part of Lemma 1, $R(x,y)$ fails to be an element of Γ_2. This means that in order to characterize the Gaussian quadrature formulas of type (1) we need more than the orthogonality and the linear multiplier representation, used so far.

6. Uniqueness

In this section we shall examine more closely the uniqueness property of (*) among a certain class of quadrature formulas.

Theorem 3. *Among all quadrature formulas of the form*

$$\iint_D f(x,y)\,dx\,dy \approx \sum_{k=1}^{n} B_k \int_{-\sqrt{1-t_k^2}}^{\sqrt{1-t_k^2}} f(t_k,y)\,dy, \qquad -1 < t_1 < \ldots < t_n < 1 \tag{10}$$

(*) is the only one, that is Gaussian.

Proof. Let (10) be a Gaussian quadrature formula. Denote $I_k^0 := I(\eta_k, 0)$ and $J_k^0 := J(t_k, 0)$. Then, by Lemma 3

$$J_k^0 \cap ((\bigcup_{l=1,l\neq k}^{n} J_l^0) \cup (\bigcup_{l=1}^{n} I_l^0)) \cap D \not\equiv \emptyset$$

and since all the lines are parallel one another, this is possible only when

$$t_k = \eta_k, \quad k = 1, \ldots, n.$$

It is easy to check, that if we apply (*) and (10) to the polynomials

$$\psi_k(x,y) = \prod_{j=1,j\neq k}^{n} (x-\eta_j)^2 \quad \in \pi_{2n-2}(\mathbb{R}^2), \quad k = 1, \ldots, n,$$

we derive that

$$A_k = B_k, \quad k = 1, \ldots, n,$$

which proves the theorem. $\square$

We next discuss quadrature formulas of the form

$$\iint_D f(x,y)\,dx\,dy \approx B_1 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} f(x_1,y)\,dy + \sum_{k=2}^{n} B_k \int_{I_k} f(x,y)\,ds, \tag{11}$$

where x_1 is fixed at the zero of $U_n(t)$ with the greatest absolute value. We prove the following theorem about the uniqueness of (*) as a Gaussian quadrature formula among the formulas of the form (11).

Theorem 4. *(*) is the only Gaussian quadrature formula among the quadrature formulas of type (11).*

Proof. We shall make use of the orthonormal basis (8) for D in $\pi_n(I\!R^2)$. Let $\mathrm{ADP}(11) = 2n-1$. Then, by Lemma 1,

$$\omega(x,y) = (x-x_1)L_{I_2}(x,y)...L_{I_n}(x,y)$$

is orthogonal to any $Q \in \pi_{n-1}(I\!R^2)$ and therefore

$$\omega(x,y) = \sum_{k=0}^{n} a_k \frac{d^k}{dx^k} U_n(x) W_k(x,y)$$

for some coefficients $\{a_k\}$. On the other hand, $\omega(x_1,y) = 0$ for each y . After comparisson of the coefficients of the univariate polynomial $\omega(x_1,y)$, we arrive at the equality

$$a_k \frac{d^k}{dx^k} U_n(x_1) = 0, \quad k = 0,\dots,n.$$

According to the fact, that x_1 is the zero of $U_n(t)$ with the greatest absolute value, by Rolle's theorem, we have that any zero t^* of $\frac{d^k}{dx^k}U_n(t),\ k = 1,\dots,n-1$ satisfies the condition

$$\mid t^* \mid < x_1.$$

Therefore

$$\frac{d^k}{dx^k} U_n(x_1) \neq 0, \quad k = 1,\dots,n,$$

which combined with the previous result gives

$$a_1 = \dots = a_n = 0.$$

Since not all a_k are zeroes, the only possibility left is when $a_0 \neq 0$. Thus

$$\omega(x,y) = a_0 U_n(x),$$

which means that the quadrature formula is of the form (10). Then, by Theorem 3, this formula is exactly (*). □

Now we introduce a new class of quadrature formulas $\Theta(n)$ that contains all ones of the type

$$\iint_D f(x,y)\,dx\,dy \approx \sum_{k=1}^{n} B_k \int_{J_k} f(x,y)\,ds, \tag{12}$$

where the set of lines $J := \{J_1,\dots,J_n\}$ satisfies the following condition: There is an index $k \in \{1,\dots,n\}$ such that all the segments $J_l \cap D,\ l =$

$1, \ldots, n,\ l \neq k$, are in one of the halfplanes, determined by J_k and there is no one in the other halfplane. For this set we have the following uniqueness theorem.

Theorem 5. *Formula (*) is the only Gaussian quadrature formula in* $\Theta(n)$.

Proof. Let (12) be a Gaussian quadrature formula. Without loss of generality, we can assume that the line, determining (12) as an element of $\Theta(n)$ is the line with equation

$$x = t_1, \quad t_1 \geq 0.$$

Let $\omega(x, y) = L_{J_1}(x, y) \ldots L_{J_n}(x, y)$. Then, by Lemma 1 and Lemma 7

$$(13) \quad \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \omega(t, y)\, dy = A\sqrt{1-t^2}U_n(t), \quad t \in (-1, 1), \quad A \in I\!R.$$

Assume, that $A = 0$. From the mean value theorem, it follows that

$$0 = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \omega(t, y)\, dy = 2\sqrt{1-t^2}\omega(M(t)), \quad t \in (-1, 1),$$
$$M(t) \in I(t, 0) \cap D.$$

In particular,

$$\omega(M(t)) = 0, \quad |\, t \,| > t_1, \quad M(t) \in I(t, 0) \cap D.$$

This is impossible, since (12) as an element of the set $\Theta(n)$ does not vanish on one of the halfplanes, determined by J_1. Therefore $A \neq 0$ and by (13), when $t = t_1$

$$U_n(t_1) = 0.$$

Let x_1 be the zero of $U_n(t)$ with the greatest absolute value. Assume $x_1 \neq t_1$ and consider (*), which by Theorem 1 is Gaussian quadrature formula. Hence, using Lemma 3, we have

$$I(x_1, 0) \cap ((\bigcup_{l=1}^{n} J_l) \cup (\bigcup_{l=2}^{n} I(x_l, 0))) \cap D \not\equiv \emptyset,$$

which can not be true, since (12) belongs to $\Theta(n)$. This implies

$$t_1 = x_1.$$

In other words, the formula (12) is of type (11) and then, by Theorem 4, (12) is actually (*). The proof is completed. □

We have to mention, that (*) has a higher degree of precision than any quadrature formula derived from the multivariate Hakopian interpolation on the unit disk D. Let us explain this more precisely.

It was shown in [1], that if $Y_1, \dots, Y_n$ are n points on the unit circle $\{(x,y) \;:\; x^2+y^2=1\}$ and $\{Y_i, Y_j\}, i=1,\dots,n, j=1,\dots,n$, are the line segments, connecting Y_i and Y_j, then for each $g(x,y) \in C(D)$ there is a unique polynomial $H_n(g;x,y) \in \pi_{n-2}(I\!R^2)$ such that

$$\int_{\{Y_i,Y_j\}} H_n(g;x,y)\,ds = \int_{\{Y_i,Y_j\}} g(x,y)\,ds, \quad 1 \le i < j \le n.$$

This polynomial can be written as

$$H_n(g;x,y) = \sum_{1\le k,l \le n} \sigma_{k,l}(x,y) \int_{\{Y_k,Y_l\}} g(x,y)\,ds,$$

where $\sigma_{k,l}(x,y) \in \pi_{n-2}(I\!R^2)$. After integrating over D, we obtain the quadrature formula

$$\text{(14)} \qquad \iint_D g(x,y)\,dx\,dy \approx \iint_D H_n(g;x,y)\,dx\,dy = \sum_{1\le k,l\le n}^{n} A_{k,l} \int_{\{Y_k,Y_l\}} g(x,y)\,ds,$$

where

$$A_{k,l} = \iint_D \sigma_{k,l}(x,y)\,dx\,dy.$$

Each quadrature of the form (14) uses integration along $N = \frac{n(n+1)}{2}$ lines and as an element of $\Theta(N)$, by Theorem 5, ADP$(14) < n(n+1)-1$. On the other hand, if we are allowed to use $N = \frac{n(n+1)}{2}$ line integrals, we would prefer (*), since it is Gaussian and thus it has a degree of precision $n(n+1)-1$. Therefore the quadrature formula (*) is better than any one of the form (14).

7. Some examples

The quadrature formula (*) can be written in the form

$$\iint_D f(x,y)\,dx\,dy \approx \sum_{k=1}^{n} A_k \int_{I_k} f,$$

where $I_k,\ k=1,\dots,n$, are the line segments in D, parallel to y axis, passing through the points $(\cos\frac{k\pi}{n+1}, 0)$ and

$$A_k = \frac{\pi}{n+1} \sin\frac{k\pi}{n+1}, \quad k=1,\dots,n.$$

Next we give examples of our formula with calculated weights and nodes.

$n = 1$

The formula is exact for all $P(x, y) \in \pi_1(I\!R^2)$.

k	Coefficients	Line segments
1	1.57079632679415	x=0.00000000000000

$n = 2$

The formula is exact for all $P(x, y) \in \pi_3(I\!R^2)$.

k	Coefficients	Line segments
1	0.90689968211791	x=0.50000000000000
2	0.90689968211700	x=-0.50000000000000

$n = 3$

The formula is exact for all $P(x, y) \in \pi_5(I\!R^2)$.

k	Coefficients	Line segments
1	0.55536036726971	x=0.70710678118655
2	0.78539816339708	x=0.00000000000000
3	0.55536036726971	x=-0.70710678118655

$n = 4$

The formula is exact for all $P(x, y) \in \pi_7(I\!R^2)$.

k	Coefficients	Line segments
1	0.36931636609870	x=0.80901699437495
2	0.59756643294895	x=0.30901699437495
3	0.59756643294804	x=-0.30901699437495
4	0.36931636609734	x=-0.80901699437495

$n = 5$

The formula is exact for all $P(x, y) \in \pi_9(I\!R^2)$.

k	Coefficients	Line segments
1	0.26179938779933	x=0.86602540378444
2	0.45344984105895	x=0.50000000000000
3	0.52359877559866	x=0.00000000000000
4	0.45344984105850	x=-0.50000000000000
5	0.26179938779842	x=-0.86602540378444

$n = 6$

The formula is exact for all $P(x, y) \in \pi_{11}(I\!R^2)$.

k	Coefficients	Line segments
1	0.19472656676044	x=0.90096886790242
2	0.35088514880954	x=0.62348980185873
3	0.43754662381298	x=0.22252093395631
4	0.43754662381298	x=-0.22252093395631
5	0.35088514880954	x=-0.62348980185873
6	1.19472656676044	x=-0.90096886790242

$n = 7$

The formula is exact for all $P(x,y) \in \pi_{13}(\mathbb{R}^2)$.

k	Coefficients	Line segments
1	0.15027943247105	x= 0.92387953251129
2	0.27768018363486	x=0.70710678118655
3	0.36280664401738	x=0.38268343236509
4	0.39269908169854	x=0.00000000000000
5	0.36280664401738	x=-0.38268343236509
6	0.27768018363486	x=-0.70710678118655
7	0.15027943247128	x=-0.92387953251129

$n = 8$

The formula is exact for all $P(x,y) \in \pi_{15}(\mathbb{R}^2)$.

k	Coefficients	Line segments
1	0.11938755218353	x= 0.93969262078591
2	0.22437520360108	x= 0.76604444311898
3	0.302299894039155	x=0.50000000000000
4	0.343762755784618	x=0.17364817766693
5	0.34376275578461	x=-0.17364817766693
6	0.30229989403870	x=-0.50000000000000
7	0.22437520360086	x=-0.76604444311898
8	0.11938755218364	x=-0.93969262078591

$n = 9$

The formula is exact for all $P(x,y) \in \pi_{17}(\mathbb{R}^2)$.

k	Coefficients	Line segments
1	0.09708055193641	x= 0.95105651629515
2	0.18465818304935	x=0.80901699437495
3	0.25416018461601	x=0.58778525229247
4	0.29878321647448	x=0.30901699437495
5	0.31415926535919	x=0.00000000000000
6	0.29878321647402	x=-0.30901699437495
7	0.25416018461556	x=-0.58778525229247
8	0.18465818304867	x=-0.80901699437495
9	0.09708055193573	x=-0.95105651629515

$n = 10$

The formula is exact for all $P(x,y) \in \pi_{17}(\mathbb{R}^2)$.

k	Coefficients	Line segments
1	0.08046263007725	x=0.95949297361450
2	0.15440665639539	x=0.84125353283118
3	0.21584157370421	x=0.65486073394529
4	0.25979029037080	x=0.41541501300189
5	0.28269234274376	x=0.14231483827329
6	0.28269234274376	x= -0.14231483827329
7	0.25979029037035	x=-0.41541501300189
8	0.21584157370398	x=-0.65486073394528
9	0.15440665639494	x=-0.84125353283118
10	0.08046263007725	x=-0.95949297361450

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